CAUSALITY CONSTRAINTS IN QUANTUM FIELD THEORY AND QUANTUM GRAVITY

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We explore the consequences of micro-causality in quantum field theory and quantum gravity. To this end, we develop methods in conformal field theories (CFT) by considering collider type experiments. Using this, we find a class of positive operators in a certain class of holographic CFTs. Positivity of these operators helps to describe the mechanism for the emergence of weakly interacting Einstein gravity description of the dual theory in anti de-Sitter spacetime. We also provide a dictionary for describing how these operators play a role in the dual bulk spacetime as shockwave states. Furthermore, we develop techniques to explore the consequences of causality in flat spacetimes by studying S-matrix consistency conditions. Using these methods, we place constraints on the higher spin spectrum of theories that include gravitational interactions.

BIOGRAPHICAL SKETCH

Nima Afkhami-Jeddi was born on Feb 14 1987 in Tehran, Iran. He moved to Toronto, Canada with his family in 2001. After high-school he briefly attended Centennial College and worked as a licenced automechanic. He then attended York University starting in 2008 majoring in biochemistry to pursue a career in the medical field. He changed career paths by switching his study major to physics and working in the atomic physics laboratory of A. Kumarakrishnan at York University. He then attended Perimeter Institute for Theoretical Physics starting in 2013 as a student in the M.Sc. program, completing his thesis under the supervision of Freddy Cachazo. Upon completion he joined Tom Hartman's group at Cornell University in 2014 resulting in this dissertation among other works. He will continue his research as a post-doctoral fellow at the Kadanoff Institute at the University of Chicago.

Dedicated to Fjolla and my family,
Thank you.

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The past five years at Cornell have been the most intellectually fulfilling period of my life so far. This would not have been possible without the guidance and support of many people with whom I have interacted throughout the years.

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CHAPTER 1

INTRODUCTION

Quantum field theories are arguably the most successful framework for describing many physical phenomena observed in nature ranging over orders of magnitudes in scale from high energy particle physics at particle accelerators to the behavior of low energy condensed matter systems. According to the modern perspective, the space of QFTs can be understood as the space of renormalization group flows describing the dynamics from high energy scales in the UV to the low energy scales in the IR. However, we typically do not have complete knowledge about the microscopic degrees of freedom and their dynamics in the UV. Moreover, we are typically interested in describing the dynamics at low energy scales which can be experimentally accessed. Therefore, instead of trying to describe the theory at all length scales, we set a more modest goal of writing down an effective field theory which accurately describes the low energy dynamics. That such a description is possible is intuitively obvious. An engineer does not need to include the details of the electronic structures of the molecules involved in order to design a bicycle and a chemist can safely ignore the strong nuclear force when describing chemical reactions. Similarly, physicists may write down effective theories that capture the physics of the system at low energy scales while ignoring certain aspects of the high energy physics that govern the dynamics at short length scales.

In the context of effective quantum field theories, one begins by identifying the low energy degrees of freedom and constructing a theory which includes all possible interactions. This simple description so far is too general to be useful. However, since we are interested in the dynamics below some cut-off energy scale Λ, by performing an expansion in $\frac{1}{\Lambda}$, we can identify and keep only the relevant interactions, thereby reducing the number of free parameters required to describe the physics up to this scale. In addition, we may use the underlying symmetries of the system to reduce the number of possible interactions. For example, Lorentz invariance seems to be satisfied at all of the energy scales that we have experimentally accessed so far and therefore the standard model, which is an effective field theory, involves only the interactions which respect this symmetry. Furthermore, we are typically interested in theories with only local interactions which further reduces the number of possible interactions. This procedure reduces the number of free parameters required to describe the low energy dynamics. However, it is not clear that this description is necessarily consistent with a theory arising from the effective low energy dynamics of a UV complete theory.

In fact as was pointed out in [1] that certain seemingly consistent low energy effective field theories respecting lorentz symmetry cannot arise from a consistent UV complete theory. They showed that obstructions to UV completion may arise from interactions that result in violations of micro-causality which are detectable at energy scales below the effective cut-off scale of the theory. In this work we will mostly focus on extracting the physical consequences of imposing causality on the effective low energy description of quantum field theories and quantum gravity.

In the recent years, it has become clear that spacetime itself must be an emergent effective low energy description of some fundamental underlying degrees of freedom in the UV. AdS/CFT [2] provides a concrete example of emergence of dynamical spacetime in anti de-Sitter (AdS) from the fundamental UV complete degrees of freedom in the dual conformal field theory (CFT). It therefore may provide a window into how certain properties of spacetime such as locality and gravitational dynamics may emerge from more fundamental degrees of freedom.

In fact, many features of gravity such as scattering amplitudes and black hole entropy can be reproduced by calculations in the supersymmetric gauge theory dual [3]. But holographic duality is likely to extend to a very large (perhaps infinite) class of CFTs, with and without supersymmetry. The theories in this "universality class" differ in their microscopic details, but all produce emergent geometry at low energies, and exhibit the rich phenomena associated to Einstein gravity in the bulk. The mechanism responsible for this remarkable universality in this class of strongly interacting quantum field theories remains to some extent mysterious. Although specific examples can be matched in great detail, there has been no universal CFT derivation of the Bekenstein-Hawking entropy formula, the universal ratio of shear viscosity to entropy density $\eta/s = 1/4\pi$ [4], or indeed any other prediction of Einstein gravity in more than three bulk dimensions — without higher curvature corrections — that is not fixed by conformal invariance or other symmetries. A universal explanation should not only match the correct answer, but also shed light on why it is independent of the microscopic details.

On the gravity side, universality is guaranteed by effective field theory. At low energies, any theory of quantum gravity is described by the Einstein action, plus small corrections suppressed by the scale M of new physics:

$$S \sim \frac{1}{G_N} \int \sqrt{-g} \left(-2\Lambda + R + \frac{c_2}{M^2} R^2 + \frac{c_3}{M^4} R^3 + \cdots \right) . \tag{1.0.1}$$

Typically the dimensionless coefficients c_i in effective field theory are order one numbers, so that the Einstein term dominates at low energies, and the theory is local below the cutoff. This suppression of the higher curvature terms is responsible for universality – it ensures, for example, that black hole entropy is Area$/4G_N$ up to small corrections. But if we choose the higher curvature terms to be ghost-free, then we can attempt to tune up the coefficients, $c_i \gg 1$. The theory is still weakly coupled due to the overall $1/G_N$. If this tuning is permissible, then universality may be violated, since the higher curvature terms modify the theory at energies much below the scale M of new particles.

At the level of 3-graviton couplings, it was shown by Camanho, Edelstein, Maldacena, and Zhiboedov (CEMZ) [5] that such tunings are in fact not allowed. They would lead to causality-violating propagation of gravitons in nontrivial backgrounds. The conclusion is that universality of the 3-graviton coupling is a fundamental property of quantum gravity, that cannot be violated even by fine tuning.

This poses a clear question for quantum field theory: Why in some class of CFTs, must the stress tensor 3-point functions $\langle TTT\rangle$ be tuned to a particular form? In a general CFT, there are three independent terms:

$$\langle TTT\rangle_{cft} = n_s\langle TTT\rangle_s + n_f\langle TTT\rangle_f + n_v\langle TTT\rangle_v, \tag{1.0.2}$$

where the $\langle TTT\rangle_i$ are known tensor structures, fixed by conformal invariance, and the n_i are coupling constants [6]. Einstein gravity predicts one particular structure, where only the overall coefficient is adjustable. Therefore, it imposes two relations on the three coupling constants n_s, n_f, and n_v [7,8]. One of these relations can be stated in terms of the Weyl anomaly coefficients as

$$a = c\ . \tag{1.0.3}$$

We showed how this particular aspect of universality arises in large-N CFTs [9]. The argument is based on a certain thought experiment that we call the holographic collider experiment. It utilizes unitarity and causality to constrain the dynamics in these theories. In [10–12] it was shown that causality of certain correlation function, in the lightcone limit, leads to the Hofman-Maldacena conformal collider bounds [13]. For example, it constrains the anomaly coefficients to lie within the window $\frac{1}{3} \leq \frac{a}{c} \leq \frac{31}{18}$. In order to derive stronger constraints such as (1.0.3) in large-N theories, we apply a similar logic, but using different kinematics which are designed to probe bulk locality. Using these bounds, we derive the universality of Einstein gravity directly from the consistency of the underlying CFT. Furthermore, we derive additional constraints on the dynamics of higher-spin particles in a UV complete theory of quantum gravity. Namely we show that higher spin particles cannot exist in isolation and that they must always exist as a part of an infinite tower of higher spin particles in the spectrum. We also generalize these bounds to flat spacetimes by considering the consistency of scattering amplitudes in the Eikonal limit. Given that the interaction length scales involved in the problem are small enough to be insensetive to the local curvature of spacetime, we conjecture that similar bounds should apply in de-Sitter and cosmological spacetimes and explore their consequences.

This thesis is organized as follows. In the first chapter we derive the dictionary betweet shockwave geometry in the AdS and certain operators in the dual CFT and elucidate the role that such geometries play in imposing constraints by introducing a bulk length operator. In the second chapter we describe the holographic collider thought experiment and use it to derive the universal behaviour of gravitational dynamics in AdS spacetimes using the consistency of the CFT dual. In the third chapter we use this technology to derive bounds on the dynamics of higher spin particles in AdS space times and generalize our results to include flat spacetimes by considering S-matrix consistency conditions.

CHAPTER 2

SHOCKWAVES FROM THE OPERATOR PRODUCT EXPANSION

Abstract[1]

We clarify and further explore the CFT dual of shockwave geometries in Anti-de Sitter. The shockwave is dual to a CFT state produced by a heavy local operator inserted at a complex point. It can also be created by light operators, smeared over complex positions. We describe the dictionary in both cases, and compare to various calculations, old and new. In the CFT, we analyze the operator product expansion in the Regge limit, and find that the leading contribution is exactly the shockwave operator, $\int du h_{uu}$, localized on a bulk geodesic. For heavy sources this is a simple consequence of conformal invariance, but for light operators it involves a smearing procedure that projects out certain double-trace contributions to the OPE. We revisit causality constraints in large-N CFT from this perspective, and show that the chaos bound in CFT coincides with a bulk condition proposed by Engelhardt and Fischetti. In particular states, this reproduces known constraints on CFT 3-point couplings.

[1]This chapter is based on N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, "Shockwaves from the Operator Product Expansion," arXiv:1709.03597 [hep-th]. I thank Simon Caron-Huot, Daliang Li, and Sasha Zhiboedov fornumerous discussions of double trace operators in the Regge limit, as well as Tom Faulkner, Diego Hofman, Jared Kaplan, Henry Maxfield, Sam McCandlish, and Arvin Moghaddam for essential conversation.

2.1 Introduction

A shockwave geometry describes the gravitational field of a point source traveling on a null geodesic [14]. In Anti-de Sitter [15–17], it plays a key role in the study of various phenomena in gravity and conformal field theory [5, 9–11, 13, 18–26]. One incarnation of this solution has the source traveling parallel to the boundary, at fixed radial distance $z = z_0$ in Poincare coordinates, as shown in figure 2.1. The resulting gravitational shock meets the boundary along a null plane, while the source itself hits the boundary only at null infinity.

In the dual CFT, this state can be created by inserting a momentum-space operator smeared against a wavepacket profile [18–20]. It can also be created by inserting a heavy local operator ψ, with dimension $\Delta_\psi \gg 1$, at a single point [10, 22, 24, 27]. Yet another proposal to create a shockwave with a different, complexified wavepacket was described in [9]. These three constructions differ in their microscopic details, but should produce all of the same observables for probes that avoid the delta function at the shockwave source.

In this chapter we will work out various aspects of the dictionary relating (linearized) AdS shockwaves to CFT operator insertions, focusing on the latter two constructions: the heavy local operator, and the complexified wavepacket. Parts of these results have been used implicitly in previous work [10, 22, 27, 28].

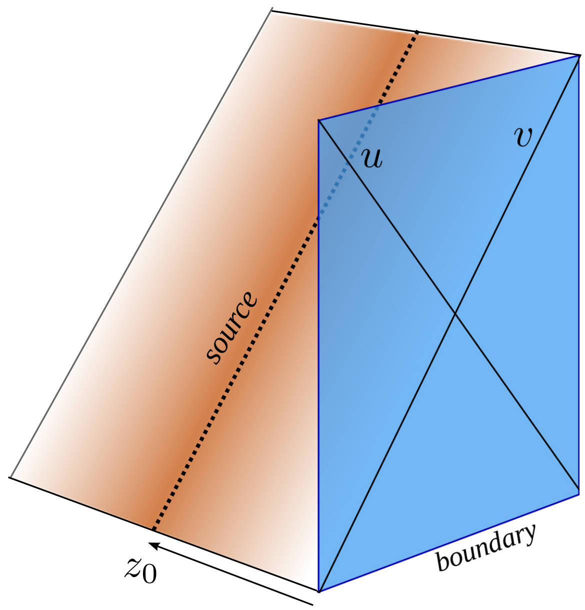


Figure 2.1: Planar shockwave (2.2.1) in Poincare coordinates. The source travels on a null geodesic parallel to the boundary, at fixed radial coordinate $z = z_0$. The shockwave is on the null plane $u = 0$.

Our main new result is a relationship between shockwaves in the bulk, and the operator product expansion (OPE) in CFT. The propagation of a probe field ϕ on the ψ-shockwave computes a four-point function, $\langle\psi\phi\phi\psi\rangle$, in a highly boosted kinematics dubbed the Regge limit [18–20,29–32]. It is well known that the crossing equation for this four-point function leads to a rich story relating double-trace anomalous dimensions in CFT to the graviton propagator in the bulk [5,18–20,25, 26,33–36]. We will essentially strip off the probes $\phi\phi$ from this story, and directly examine the $\psi\psi$ OPE in the Regge limit. The leading term is precisely the bulk shockwave operator.

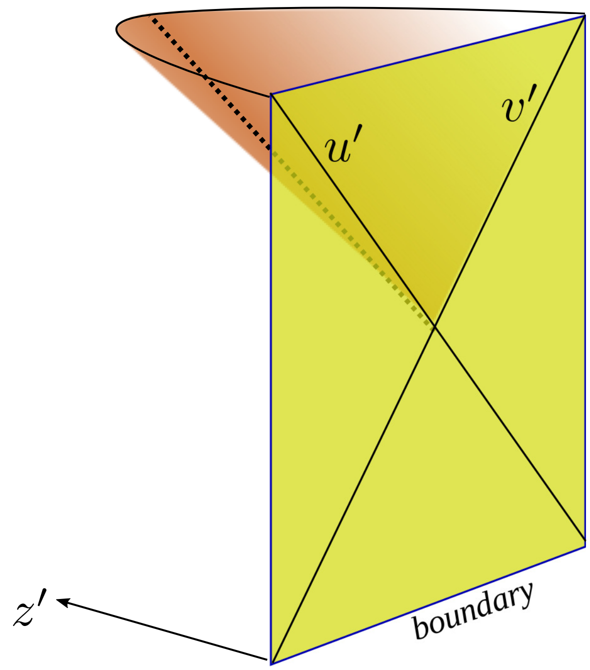


Figure 2.2: Spherical shockwave (2.2.7) in Poincare coordinates. The source travels on a radial null geodesic that hits the boundary at the origin, $u' = v' = 0$. The resulting shockwave is a null cone.

Of course, the correlators and the OPE ultimately contain the same information, but the OPE point of view is useful for several reasons. First, analogous results in the lightcone limit were a crucial step toward deriving the averaged null energy condition [10] and quantum null energy condition [37] from causality. Second, it makes clear why various bulk and boundary results must match, without actually doing the calculations, and extends this match — including causality constraints — to a more general class of quantum states. Finally, it generalizes existing constraints to higher-point functions and multiple shocks, though we will not explore this in detail here.

In addition to the Regge OPE, our other main result is a technical improvement on our previous work on large-N causality constraints [9]. We use conformal Regge theory to confirm an assumption made there regarding the contributions of double-trace operators to smeared, spinning conformal correlators, in the shockwave regime. This is a natural extension of the Regge OPE formula to spinning operators.

In the rest of this introduction we give a brief summary of the main results.

2.1.1 Shockwaves from heavy operators

The simplest way to create a shockwave is by inserting a heavy scalar primary operator ψ, since this does not require any smearing. It does, however, require somewhat exotic kinematics, with operators inserted near $i\infty$. For example, the shockwave depicted in figure 2.1, with a real metric (see (2.2.1)), is dual to the CFT state

$$|\Psi\rangle = \psi(u = \frac{i\Delta_\psi}{2E}, v = \frac{2iz_0^2 E}{\Delta_\psi})|0\rangle \ , \tag{2.1.1}$$

where (u, v) are null coordinates in Minkowski, E is the energy of the shockwave, and the duality holds for $1 \ll \Delta_\psi \ll E$. Analogous results apply to the spherical shockwave in figure 2.2, which is created by a local operator inserted near the origin, and is related to the planar shockwave by a null inversion. We start in section 2.2 by deriving this dictionary and comparing the stress tensor one-point functions to CFT.

Next, in section 2.3, we turn to the relationship between shockwaves and the OPE. In the lightcone limit, the leading term in the OPE is the averaged null energy operator $\int duT_{uu}$ [10]. In the Regge limit, we will show that the leading contribution is

$$\psi(-u,-v)\psi(u,v) \ \propto \ 1 - iEz_0^2 \int_\gamma du h_{uu} \tag{2.1.2}$$

as $u \to \infty$ with uv fixed. The "shockwave operator" on the right-hand side is the bulk metric perturbation integrated over a null geodesic through AdS_{D+1}, with $F \geq 3$. It can also be viewed as the linearized length operator. Physically, it corresponds to the source for a planar shockwave, or to the time delay in a background produced by other operators.

Although (2.1.2) looks like a bulk formula, it can be interpreted as a CFT operator by translating h_{uu} into CFT language. In the lightcone limit, $h_{uu} \sim T_{uu}$, so this reproduces the averaged null energy operator derived in [10]. More generally, the translation to CFT follows from the HKLL formula [38] for h_{uu}. Interpreted this way, and keeping only the leading single-trace term, (2.1.2) gives a formula for the stress tensor contribution in the Regge limit, in any CFT. In CFT language, this operator is the part of the OPE block [39, 40] that grows in the Regge limit. In a holographic theory, (2.1.2) contains more information than the OPE block, since it also accounts for multitrace contributions.

2.1.2 Shockwaves from light operators

The above discussion applies only to operators with $\Delta_\psi \gg 1$. For a light operator O — and in particular, for the stress tensor $T_{\mu\nu}$ which will be important for the discussion of causality — the OPE is not given by (2.1.2). In gravity language, a light operator inserted at a point creates a wave that spreads, not a particle that travels on a geodesic. To remedy this, we insert a wavepacket. The smeared operators behave just like heavy operators, in that they create a state dual to the bulk shockwave, and have exactly the same Regge OPE (2.1.2), with (u, v) now interpreted as the wavepacket centers. Importantly, this OPE has no contributions from the O^2 double-trace operators that would appear in the unsmeared OPE. We will derive this statement from the bulk by smearing a Witten diagram vertex, and from CFT using conformal Regge theory [32]. It applies to both scalar and spinning operators O.

In the scalar case, the smeared operators $\tilde{O}$ have exactly the same Regge OPE as the heavy operator ψ in (2.1.2). For spinning operators, rather than deriving an OPE formula like (2.1.2) explicitly, we work directly with the four-point function, and show that smearing projects out the double-trace contributions to the conformal Regge amplitude. The procedure is based closely on [25, 26], and differs only by the fact that our wavepackets are rotated into a complex direction, following [9]. This choice is motivated by the chaos bound [41], which applies only in a kinematic regime where certain points are spacelike separated, and cannot be applied to ordinary wavepackets smeared over real spacetime points. (Other arguments, rather than the chaos bound, were used to derive causality constraints from real wavepackets in [25, 26]; see also [42].)

Exactly the same complex smearing procedure (up to a conformal transformation) was used in our previous work on large-N causality constraints [9]. There, we suggested on other grounds that the double-trace contributions should drop out, but did not show it explicitly; the present result confirms this, and therefore closes this potential loophole in the argument.

2.1.3 Causality constraints

In a CFT with a gravity-like holographic dual, the shockwave operator in (2.1.2) (including multitraces) dominates the 4-point function, so it is subject to the causality constraints on conformal correlators derived in [9,41]. Therefore, $\int_\gamma du h_{uu}$ has a positive expectation value in states defined perturbatively about the vacuum. This generalizes the averaged null energy condition (ANEC) $\int du T_{uu} \geq 0$, and reduces to the ANEC in a lightcone limit.

The ANEC has also been derived from the monotonicity of relative entropy [37]. It would be very interesting to understand the Regge analogue of that calculation, or more generally, the constraints from quantum information in the Regge limit.

If the state has a geometric dual, then the sign constraint $\int_\gamma du h_{uu} \geq 0$ coincides with the bulk causality condition of Engelhardt and Fischetti [43]. Noteably, it is apparently weaker than the averaged null curvature condition $\int du R_{uu} \geq 0$, or bulk null energy condition, which were the starting point for Gao and Wald [44] to prove that asymptotically AdS spacetimes satisfy boundary causality. Our constraint also applies to non-geometric states, such as superpositions of different geometries.

We postpone a detailed discussion of causality constraints to section 2.4, but the brief version is that this inequality encompasses, and generalizes, a number of known causality constraints in CFT: the scalar causality constraints derived in [10], the Hofman-Maldacena bounds [11–13, 21], the averaged null energy condition [10, 45, 46], and the $a = c$ type bounds derived on the gravity side in [5] and the CFT side in [9,25,26]. In holographic CFTs, the result here is more general, since we will show that the operator $\int_\gamma du h_{uu}$ is positive in a wider class of states. It can be evaluated in particular states to reproduce each of these previous constraints.

2.2 Shockwaves from Heavy Operators

In this section we review the shockwave metric and the CFT construction using a heavy local operator insertion. Most of these results are known in some form, but we will start from the beginning and clarify a few points along the way. The main goal is to derive the dictionary (2.1.1) for the planar shock. This result holds in general dimensions, AdS_{d+1} with $D \geq 2$.

To summarize briefly, it is easiest to start with the dictionary for the spherical shockwave [15, 16], depicted in figure 2.2. In this geometry, the source particle hits the boundary at the origin, $u' = v' = 0$, so it is roughly dual to the CFT state with an operator inserted there, $\psi(0)|0\rangle$. 'Roughly' because such a state is not normalizable, so it is useful to regulate it by moving the source slightly into Euclidean time, inserting the operator at $t = i\delta$. In the bulk, this corresponds to a geometry where the source does not quite hit the boundary, but has closest approach $z = \delta$. In the limit $\delta \to 0$, it becomes the shockwave geometry.

The planar shockwave (figure 2.1) and the spherical shockwave (figure 2.2) are related by a null inversion. This is a conformal transformation that sends $v' \to -1/v$. The effect is easiest to understand on the Minkowski cylinder; see figure 2.3. The original (u', v') coordinates cover a diamond on the cylinder. The null inversion sends $v' = 0 \to v = -\infty$, so the new (u, v) coordinates cover a new diamond, shifted in the null direction by 'half of a patch'.

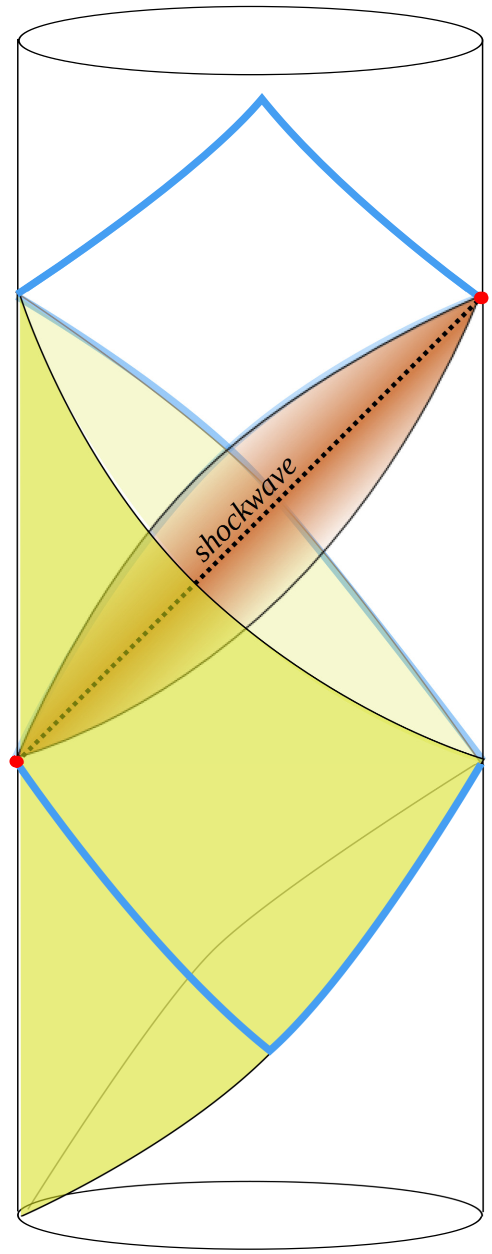


Figure 2.3: Shockwave in global AdS, where the boundary is the Lorentzian cylinder. The dashed line is the source, which hits the boundary at the red dots. This source creates a shockwave on the shaded null surface. In the shaded-yellow Poincare patch, the solution is the spherical shockwave. In the shifted Poincare patch, shown as a thick blue outline, the solution is the planar shockwave. The yellow and blue patches in this figure correspond to the same color-coding as the other figures.

Since the original insertion was at $v = 0$, the operator insertion for a planar shockwave should be roughly at $-\infty$. Carefully keeping track of the Euclidean-time regulator gives the precise statement of the dictionary, (2.1.1).

We will derive these results in the opposite order, starting with the planar shockwave, then performing the null inversion to reproduce the spherical shockwave.

2.2.1 Shockwave solutions in AdS

Planar coordinates

The metric of the planar shockwave in AdS_{d+1} [15,16] is

$$ds^2 = \frac{L^2}{z^2}(-dudv + dz^2 + d\vec{x}^2) + h_{uu}^{Shock} du^2 \tag{2.2.1}$$

with

$$h_{uu}^{Shock} = E\frac{16\pi G_N (4\pi)^{\frac{1-d}{2}}\Gamma\left(\frac{d+1}{2}\right) z_0}{zL^{d-1}d(d-1)}\left(\frac{\rho^2}{1-\rho^2}\right)^{1-d} {}_2F_1\left(d-1, \frac{d+1}{2}; d+1; 1-\frac{1}{\rho^2}\right)\delta(u) \ . \tag{2.2.2}$$

We will assume that the source is localized at $u = 0, z = z_0, \vec{x} = 0$ and hence ρ is given by [5]

$$\rho = \sqrt{\frac{(z-z_0)^2 + \vec{x}^2}{(z+z_0)^2 + \vec{x}^2}} \ . \tag{2.2.3}$$

This geometry is illustrated in figure 2.1. The source travels on a null geodesic at fixed radial distance, $z = z_0$, and the geometry solves the Einstein equation with this source, which reduces to

$$z^{d-1}\partial_z\left[z^{1-d}\partial_z(z^2h^{Shock}_{uu})\right]+z^2\vec{\partial}^2h^{Shock}_{uu}=-16\pi G_N E z_0^{d-1}\delta(u)\delta^{d-2}(\vec{x})\delta(z-z_0)\ . \tag{2.2.4}$$

Holographic stress tensor

According to the usual holographic dictionary, the boundary stress tensor is proportional to the $O(z^{d-2})$ components of the metric [47–49]. For the planar shockwave (2.2.1), this gives

$$\langle T_{uu}(u,v,\vec{x})\rangle = E\frac{2^{d-2}\ \Gamma\left(\frac{d-1}{2}\right)z_0^d}{\pi^{\frac{d-1}{2}}(z_0^2+\vec{x}^2)^{d-1}}\delta(u)\ . \tag{2.2.5}$$

Other components vanish. Integrating gives the total energy, E.

Spherical coordinates

The spherical shockwave can be obtained from (2.2.1) by a coordinate change [15, 16]:

$$u=u'-\frac{\vec{x}'^2}{v'}-\frac{z'^2}{v'}\ ,\quad v=-\frac{z_0^2}{v'}\ ,\quad z=\frac{z'z_0}{v'}\ ,\quad x_i=\frac{x_i'z_0}{v'}\ . \tag{2.2.6}$$

This acts as a conformal transformation on the boundary which we refer to as the null inversion. If we write $u'=t'-y'$, $v'=t'+y'$, and $r^2=y'^2+\vec{x}'^2$, then under this transformation, the metric (2.2.1) becomes

$$ds^2=\frac{L^2}{z'^2}\left(dz'^2-dt'^2+dr^2+r^2d\Omega^2\right)+ds_p^2\ , \tag{2.2.7}$$

where the shockwave part, coming from h_{uu} in (2.2.1), is

$$ds_p^2=\frac{L^2}{z'^2}\left(\frac{2tz'}{z_0}\right)f\left(\frac{t'-z'}{z'}\right)\delta\left(t'-\sqrt{r^2+z'^2}\right)d\left(t'-\sqrt{r^2+z'^2}\right)^2 \tag{2.2.8}$$

with

$$f\left(\frac{t'-z'}{z'}\right) = \frac{16\pi G_N E \pi^{\frac{1-d}{2}} \Gamma\left(\frac{d+1}{2}\right) z_0}{L^{d-3}(d-1)d} \left(\frac{t'-z'}{z'}\right)^{1-d} {}_2F_1\left(d-1, \frac{d+1}{2}; d+1; \frac{2z'}{z'-t'}\right) . \tag{2.2.9}$$

This solution is shown in figure 2.2. The source particle travels radially, on a null geodesic that hits the boundary at the origin, and the resulting shockwave is an expanding spherical shell on the boundary. The metric (2.2.7) can also be obtained from an infinitely boosted AdS-Schwarzschild black hole, where the mass of the black hole is scaled down to hold the total energy fixed.[2]

The boundary stress tensor obtained from (2.2.7) is supported on the null cone,

$$\langle T_{\bar{u}\bar{u}} \rangle = E \frac{\pi^{\frac{1}{2}-\frac{d}{2}}\ \Gamma\left(\frac{d-1}{2}\right)}{r^{d-2}} \delta(\bar{u}) \ , \tag{2.2.11}$$

where $\bar{u} = t' - r$. This shockwave has total energy $2E$, twice that of the planar shockwave, because part of the expanding spherical shell lies at null infinity in the planar coordinates and was not included there.

Global picture

The important thing to notice in the coordinate change (2.2.6) is that it involves the null inversion $v \sim -1/v'$. This maps the origin in the primed coordinates to I^- in the unprimed coordinates. This makes sense, since the source particle hits the boundary at the origin of the spherical shock, but only at null infinity in the

[2]The coordinate change for comparison to [16] is

$$y_+ = \frac{t'^2 - z'^2 - r^2}{z' + t'} \ , \qquad y_- = -\frac{L^2}{z'+t'} \ , \qquad \sum_{i=2}^{d} y_i^2 = \frac{rL}{z'+t'} \ . \tag{2.2.10}$$

planar shock. The shockwave solution in global coordinates is illustrated in figure 2.3. The (u, v) and (u', v') coordinates cover two different Poincare patches, related by a null shift.

2.2.2 Shockwave states in CFT

Now we will describe how to create these states in CFT. As mentioned in the introduction, there are a number of different ways to do this, which all produce the same physics away from the point-particle source. There is also the choice of whether we want to reproduce expectation values in the 'in-in' or 'in-out' sense. We will start with the simplest and most intuitive case: in-in correlators in the spherical shockwave created by a heavy local operator. This state has also been discussed in [10, 11, 22, 24]. Much of this applies to any CFT, whether or not it is holographic, so for now the discussion is general and we will specialize to holographic CFTs later. We restrict to $D > 2$.

Insertion at the origin

A pure state in CFT can be created by inserting a local operator, $\psi(x)|0\rangle$. (We will always take space in the CFT to be R^{d-1}, so the spectrum is continuous.) If x is a point in the Lorentzian spacetime, this state is not normalizable, but an operator insertion at non-zero Euclidean time produces a normalizable state. We first consider a shockwave state created by inserting ψ at $y' = 0$, $t' = i\delta$ and $\vec{x}' = 0$, with δ small. In null coordinates $v' = t' + y'$ and $u' = t' - y'$, with the metric

$ds^2 = -du'dv' + d\vec{x}'^2$, we have

$$|\psi_0\rangle = \psi(x_0)|0\rangle \ , \qquad \langle\psi_0| = \langle 0|\psi(x_0^*) \ . \tag{2.2.12}$$

where

$$x_0 = (u_0', v_0', \vec{x}_0') = (i\delta, i\delta, \vec{0}) \ , \qquad x_0^* = (-i\delta, -i\delta, \vec{0}) \ . \tag{2.2.13}$$

The subscript 0 is a reminder that ψ is inserted near the origin, since another choice is considered below. The conjugate state in (2.2.12) is defined with respect to the usual Hermitian conjugate in Minkowski space, which does nothing to real operators inserted at real Minkowski points, but conjugates the coordinates when they are complex, in particular reflecting Euclidean time $it \to -it$.

Now let's compute the stress tensor expectation value in the state $|\psi_0\rangle$. The three-point function of the stress tensor and two scalars, in D spacetime dimensions, is entirely fixed by conformal invariance [6],

$$\langle T_{\mu\nu}(x_1)\psi(x_2)\psi(x_3)\rangle = \frac{a_d}{x_{12}^d x_{23}^{2\Delta_\psi - d} x_{31}^d}\left(\frac{X_\mu X_\nu}{X^2} - \frac{\eta_{\mu\nu}}{d}\right) \ , \tag{2.2.14}$$

where

$$x_{IJ} = |x_I - x_J| \ , \qquad X^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{12}^\mu}{x_{12}^2} \ , \qquad X^2 = \frac{x_{23}^2}{x_{13}^2 x_{12}^2} \tag{2.2.15}$$

and the Ward identity fixes

$$a_d = -\Delta_\psi \frac{\Gamma(d/2)d}{2\pi^{d/2}(d-1)} \ . \tag{2.2.16}$$

We have normalized the scalar by $\langle\psi(x_1)\psi(x_2)\rangle = |x_1 - x_2|^{-2\Delta_\psi}$, and the stress tensor is canonically normalized (for example by the Noether procedure).

This can be used to calculate the expectation value

$$\langle T_{\mu\nu}(x)\rangle = \frac{\langle\psi_0|T_{\mu\nu}(x)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle} = \frac{\langle\psi(x_0^*)T_{\mu\nu}(u, v, \vec{x})\psi(x_0)\rangle}{\langle\psi(x_0^*)\psi(x_0)\rangle} \ . \tag{2.2.17}$$

Using (3.4.4), this formula, at finite δ, agrees exactly with the boundary stress tensor produced by a boosted AdS-Schwarzschild black hole, at finite boost [16].[3] To take the limit $\delta \to 0$, note that

$$\lim_{\delta\to 0} \frac{\delta^{n+1}}{(z^2+\delta^2)^{1+\frac{n}{2}}} = \frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\delta(z) \ . \tag{2.2.18}$$

Therefore

$$\langle T_{\bar{u}\bar{u}}\rangle = \frac{\Delta_\psi \Gamma\left(\frac{d-1}{2}\right)}{2\pi^{\frac{d-1}{2}} r^{d-2}\delta}\delta(\bar{u}) \ . \tag{2.2.19}$$

where $\bar{u} = t' - r$, $\bar{v} = t' + r$, and $r^2 = y'^2 + \vec{x}'^2$. Thus the stress-energy is supported on a lightcone emanating from the origin. This stress tensor exactly matches with the boundary stress tensor (2.2.11) computed for the AdS spherical shockwave (2.2.7) once we identify

$$E = \frac{\Delta_\psi}{2\delta} \ . \tag{2.2.20}$$

This calculation did not assume holography or any particular Δ_ψ, but we will see later that in a holographic theory, this state is dual to a localized shockwave in the bulk when $c_T \gg \Delta_\psi \gg 1$, where $c_T \sim N^2$ is the coefficient in $\langle TT\rangle$.

---

[3] Explicitly, for $\vec{x}' = 0$ (and dropping the primes),

$$\langle T_{uu}(u,v,\vec{x}=0)\rangle = -\frac{2^{d-2}a_d\delta^d}{(u^2+\delta^2)^{1+\frac{d}{2}}(v^2+\delta^2)^{-1+\frac{d}{2}}} \ , \quad \langle T_{vv}(u,v,\vec{x}=0)\rangle = -\frac{2^{d-2}a_d\delta^d}{(u^2+\delta^2)^{-1+\frac{d}{2}}(v^2+\delta^2)^{1+\frac{d}{2}}} \ ,$$

$$\langle T_{uv}(u,v,\vec{x}=0)\rangle = -\frac{2^{d-2}a_d\delta^d(d-2)}{d(u^2+\delta^2)^{d/2}(v^2+\delta^2)^{d/2}} \ , \qquad \langle T_{ij}(u,v,\vec{x}=0)\rangle = -\frac{2^d a_d\delta^d\delta_{ij}}{d(u^2+\delta^2)^{d/2}(v^2+\delta^2)^{d/2}} \ .$$

Insertion at infinity

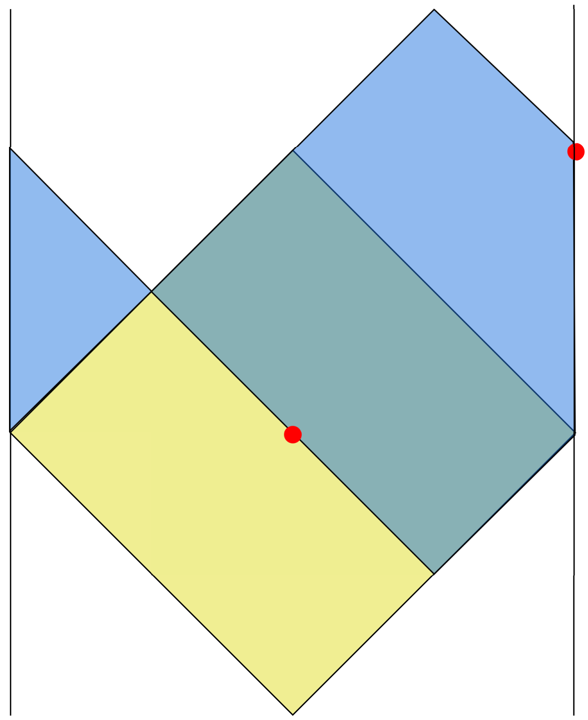


Figure 2.4: Boundary version of figure 2.3, showing the action of the null inversion (2.2.21) on Minkowski patches. The left and right sides of the diagram are identified to make the Lorentzian cylinder. The (u', v') coordinates cover the Minkowski spacetime shown as a yellow diamond. The coordinates (u, v) cover the shifted patch, shown as a blue diamond. These overlap in the region $v' > 0$, $v < 0$. The shockwave hits the boundary at the red dots; it is created at the origin of the yellow diamond, which is on I^- of the blue diamond.

So far we have been working in coordinates where the shockwave operator is inserted near the origin of Minkowski space. Mapping the CFT to a Lorentzian cylinder, this Minkowski space covers just one patch — the shaded yellow patch in figure 2.2. Let us now shift to the next patch using the null inversion [13]:

$$v \to -\frac{z_0^2}{v}, \quad u \to u - \frac{\vec{x}^{\,2}}{v}, \quad x^i \to \frac{z_0 x^i}{v} \;, \tag{2.2.21}$$

where z_0 is some length scale. This is the conformal transformation induced on the boundary by the AdS coordinate change (2.2.6). The old and new Minkowski patches are shown on the cylinder in fig. 2.4.

The origin in the old patch is a point on past null infinity in the new patch. Therefore, in these coordinates, we have a shockwave state $|\psi_\infty\rangle$ created by insertion of a local scalar operator near infinity:

$$|\psi_\infty\rangle = \psi(x_0)|0\rangle \ , \qquad \langle\psi_\infty| = \langle 0|\psi(x_0^*) \ , \tag{2.2.22}$$

where now

$$x_0 = (u_0, v_0, \vec{x}_0) = (i\delta, \frac{iz_0^2}{\delta}, \vec{0}) \ , \qquad x_0^* = (-i\delta, -\frac{iz_0^2}{\delta}, \vec{0}) \ . \tag{2.2.23}$$

Note that $\langle\psi_\infty|\psi_\infty\rangle = (4z_0^2)^{-2\Delta_\psi}$. Again, the Hermitian conjugate is the standard one acting on states in Minkowski space.

This is the dictionary for the planar shockwave quoted in the introduction. Note that the stress tensor is real, as it must be, since it is an expectation value in the 'in-in' sense.

To find the stress tensor, we can either apply (3.4.4) directly to the new kinematics, or apply the null inversion to (2.2.19). The result agrees with the planar shockwave (2.2.5) with energy $E = \Delta_\psi/(2\delta)$ in the limit $\delta \to 0$.

Cylinder picture

Although it may seem strange to insert a local operator at $v \sim i\infty$, it is natural when viewed on the global cylinder in figure 2.3 or 2.4. To see this, set $\vec{x} = 0$ and $z_0 = 1$. The map from the plane to the cylinder (in the sense of the Poincare patch embedding) is[4]

$$u = \frac{1}{\tan\frac{\theta-\tau}{2}}, \qquad v = -\frac{1}{\tan\frac{\theta+\tau}{2}} \ , \tag{2.2.25}$$

where the cylinder coordinates are $-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2$. The null inversion, $v' = -1/v$, translates along the cylinder in the null direction:

$$\theta' = \theta + \frac{\pi}{2}, \quad \tau' = \tau + \frac{\pi}{2} \ . \tag{2.2.26}$$

To create the spherical shockwave, we insert ψ at $u = v = i\delta \ll 1$, *i.e.*,

$$\theta = \pi \ , \qquad \tau = 2i\delta \ . \tag{2.2.27}$$

The planar shockwave (2.2.23) is inserted at

$$\theta = \frac{\pi}{2}, \qquad \tau = -\frac{\pi}{2} + 2i\delta \ . \tag{2.2.28}$$

So we see that the insertion at $v = i\infty$ simply corresponds to a small Euclidean-time offset on the cylinder.

Notice also that (2.2.25) is invariant under $(\tau,\theta) \to (\tau+\pi, \theta+\pi)$. In the (u,v) patch, this takes points on I^- to points on I^+. This will have interesting implications for positivity conditions in CFT with timelike separated points, discussed in section 2.5.1.

2.3 The Regge OPE

In the previous section, we found that the spherical and planar shockwaves in AdS have exactly the same $\langle T_{\mu\nu}\rangle$ as the CFT states $|\psi_0\rangle$ and $|\psi_\infty\rangle$, respectively. This is

[4] Including the transverse directions,

$$u, v = \frac{\sin\tau \mp \cos\phi\sin\theta}{\cos\tau - \cos\theta} \ , \quad x_i = \frac{\sin\theta\sin\phi\Omega_i}{\cos\tau - \cos\theta} \tag{2.2.24}$$

where $\Omega_{i=1\ldots d-2}$ with $\Omega_i^2 = 1$ are coordinates on a unit S^{d-3}. Above we set $\phi = \pi$.

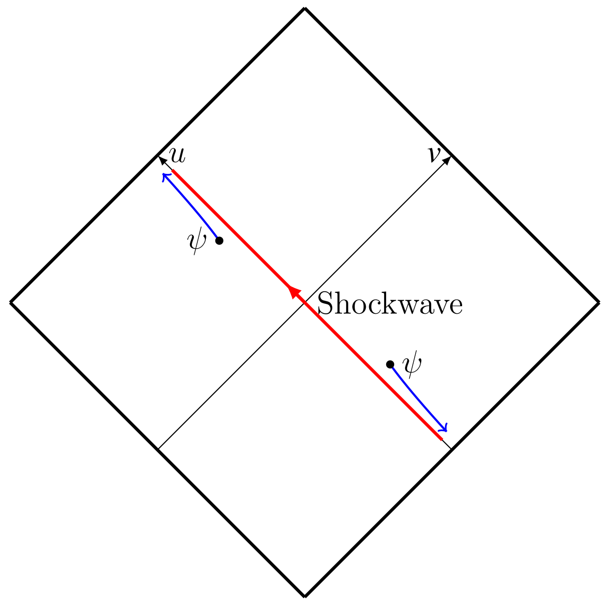


Figure 2.5: The Regge limit: the operator product $\psi(u,v)\psi(-u,-v)$ can be replaced by a shockwave propagating along $v=0$ when $\Delta_\psi \gg 1$.

a consequence of conformal symmetry that holds in any CFT; we have not yet used holographic duality. To claim that the states are dual requires n-point functions to match in this state as well. In a holographic CFT, finding the correct 1-point functions for single trace operators is enough, since these fully determine the bulk geometry, so higher-point functions are guaranteed to match. We will work it out explicitly to see how the shockwave naturally comes out of the OPE on the CFT side.

Consider a scalar operator ψ in the CFT, inserted symmetrically about the origin: $\psi(u,v)\psi(-u,-v)$, with $v < 0 < u$. The Regge limit [18–20, 29–32] of the OPE is defined by sending

$$v \to 0, \quad u \to \infty, \quad uv = \text{fixed} . \tag{2.3.1}$$

See figure 2.5. This limit is usually discussed inside 4-point functions, but it also makes sense within the OPE, assuming that any other operator positions are held fixed at finite values as $v \to 0$. We will first derive the OPE on the gravity side, then compare to CFT. We assume that ψ is a heavy probe operator, meaning $c_T \gg \Delta_\psi \gg 1$, where $c_T \sim 1/G_N$ is the coefficient of the stress tensor two-point function. The restriction to large Δ_ψ ensures that we do not need to worry about the ψ 1-point function, so that the states discussed in section 2.2 are indeed dual to localized shockwaves.

(Caveat: The roles of u and v are swapped in this section compared to section 2.2. Eventually we will need to discuss shockwaves going in both directions, so this is unavoidable.)

2.3.1 The length operator

On the gravity side, the two-point function of a heavy probe can be computed in the WKB approximation. Assuming the bulk field dual to ψ does not interact with any background fields that are turned on, it is given by the geodesic length connecting the two insertions,

$$\langle \Psi | \psi(x_1)\psi(x_2) | \Psi \rangle = e^{-\Delta_\psi L(x_1,x_2)} \ . \tag{2.3.2}$$

This holds in any state $|\Psi\rangle$ with a geometric dual, so we can attempt to write this as an operator relation:

$$\psi(x_1)\psi(x_2) = e^{-\Delta_\psi L(x_1,x_2)} \ , \tag{2.3.3}$$

where now L is an operator built from the bulk metric. This is not a true operator equation, but holds at least in semiclassical states. The same relation has been exploited recently in the $D = 2$ context [50, 51], but here we restrict to $D \geq 3$.

By expanding (2.3.3) perturbatively, we can turn this into a simple OPE. Choose $x_1 = -x_2 = (u, v, \vec{0})$, and write the bulk metric as

$$g_{\mu\nu} = g_{\mu\nu}^{AdS} + h_{\mu\nu} \ . \tag{2.3.4}$$

In pure AdS, the geodesic that connects x_1 and x_2 is given by

$$v'(u') = \frac{u'v}{u} \ , \qquad z'(u') = \sqrt{\frac{(u'^2 - u^2)v}{u}} \ , \qquad \vec{x}'(u') = \vec{0} \ . \tag{2.3.5}$$

Now expanding the length operator to linear order in $h_{\mu\nu}$ yields

$$\frac{\psi(u,v)\psi(-u,-v)}{\langle\psi(u,v)\psi(-u,-v)\rangle} = 1 - \Delta_\psi \int_{-u}^{u} du' \frac{(u^2 - u'^2)}{2u^3} \left(u^2 h_{uu} + 2uv h_{uv} + v^2 h_{vv}\right) + \mathcal{O}(h^2) \ , \tag{2.3.6}$$

where $h_{\mu\nu} = h_{\mu\nu}(u', v'(u'), z'(u'), \vec{x}'(u'))$. In the Regge limit (2.3.1) this becomes

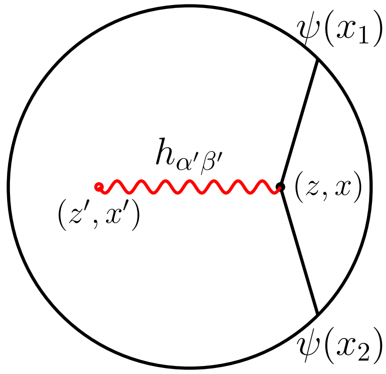
$$\frac{\psi(u,v)\psi(-u,-v)}{\langle\psi(u,v)\psi(-u,-v)\rangle} = 1 - \frac{\Delta_\psi u}{2} \int_{-\infty}^{\infty} du' h_{uu}(u', v' = 0, z' = \sqrt{-uv}, \vec{0}) \ . \tag{2.3.7}$$

This is the Regge OPE, written in gravity language, and is one of our main results in the simplest setting of heavy operator insertions. The integral is over a null geodesic parallel to the boundary, *i.e.,* the source for a planar shockwave, so we refer to the $O(h)$ correction as a shockwave operator. By exchanging Δ_ψ for E, the shockwave energy, it can also be written in the form (2.1.2).

We will make a few remarks on the bulk interpretation of (2.3.7), then turn to the CFT and examine its operator content.

From Witten diagrams

The same formula (2.3.7) can be derived from Witten diagrams. Consider the scalar-scalar-graviton vertex diagram in AdS:



(2.3.8)

In the Regge limit, with $\Delta_\psi \gg 1$, this diagram reduces to

$$\Pi_{\alpha'\beta'}(x_1,x_2;z',x') = -\frac{\Delta_\psi u}{2}\int_{-\infty}^{\infty} du'' G_{uu\alpha'\beta'}(u'', v=0, \vec{x}=0, z=\sqrt{-uv}; z', x') \tag{2.3.9}$$

where $G_{\alpha\beta\alpha'\beta'}$ is the graviton propagator in the bulk. This indicates that the full vertex is dominated by a single geodesic Witten diagram [52]; the geodesic becomes null in the Regge limit, so it can be viewed as the source for a shockwave. The derivation of (2.3.9) is in appendix A.1.

The vertex result (2.3.9) is equivalent to the OPE statement (2.3.7). To see this, simply insert both equations into a Witten diagram.

Relation to imaginary shockwaves

Inserting the Regge OPE (2.3.7) into a correlation function shows that we can replace the ψ operators by a linearized shockwave:

$$\frac{\langle \psi(x_1) O_3(x_3) O_4(x_4)\cdots O_n(x_n)\psi(x_2)\rangle}{\langle \psi(x_1)\psi(x_2)\rangle} \approx \langle O_3(x_3)O_4(x_4)\cdots O_n(x_n)\rangle_{shock} \tag{2.3.10}$$

Here the O's are primary operators, possibly with spin, obeying $\Delta_O \ll c_T$ so that we can work perturbatively. (Operator ordering is discussed below.) On the right-hand side is the $(n-2)$-point function in a shockwave background, with metric

$$ds^2 = \frac{1}{z'^2}(-du'dv' + dz'^2 + d\vec{x}'^2) + h_{vv}^{Shock} dv'^2 \tag{2.3.11}$$

where h_{vv}^{Shock} is the shockwave metric with $z_0 = \sqrt{-uv}$ and imaginary energy:

$$E = \frac{i\Delta_\psi}{2v} \ . \tag{2.3.12}$$

(This shockwave is oriented in the opposite direction as (2.2.2), so $u \to v'$ in that formula, and we have set $L = 1$.)

Note the crucial factor of i: If the ψ's are inserted at real points (u, v) in Minkowski spacetime, then the correlator (2.3.10) is computed in a purely imaginary shockwave. The i can also be seen by computing $\langle \psi(u, v) T_{\mu\nu}(0) \psi(-u, -v) \rangle$, which is purely imaginary for real u, v, indicating that the bulk metric perturbation is also imaginary. This is related to the discussion in section 2.2, where we found that the real shockwave corresponds to operators inserted at imaginary (u, v).

In terms of Witten diagrams, the same conclusions follow from doing the u'' integral in (2.3.9), which gives the shockwave metric:

$$\Pi_{\alpha'\beta'}(x_1, x_2; z', x') = \tfrac{1}{2} \langle \psi(u, v) \psi(-u, -v) \rangle h_{vv}^{Shock} \delta_{\alpha'}^{v} \delta_{\beta'}^{v} \ . \tag{2.3.13}$$

Real shockwaves

The OPE, and the result (2.3.10), also apply to $u \to i\infty$ as in the shockwave state $|\psi_\infty\rangle$ discussed in section 2.2. This is the limit that must be taken in order to reproduce correlation functions in the real shockwave geometry, *i.e.,* the metric (2.2.2) with $u \to v'$ and real $E = \Delta_\psi/(2\delta)$. This confirms, as expected, that higher-point functions in the shockwave state $|\psi_\infty\rangle$ indeed agree with gravity calculations on the real shockwave background.

Operator ordering

The operator ordering in (2.3.7) is encoded in the choice of contour for the u'-integral. This is entirely analogous to the lightcone limit, discussed in detail in section 3 of [10]. Effectively, the u' integral circles poles coming from operators inserted to one side of the $\psi\psi$ insertion in the correlators, and avoids poles from operators on the other side. We will work out some explicit examples in section 2.4.2.

2.3.2 CFT interpretation

The Regge OPE (2.3.7) is written in terms of the bulk metric. Next we want to interpret it in terms of CFT operators. At this point, we need to make the distinction between holographic and non-holographic CFTs. In any CFT, the stress tensor conformal block grows in the Regge limit, and we will see below that this growth is captured exactly by the shockwave operator (2.3.7). But this is not necessarily useful, because in general CFTs, the OPE cannot be used to calculate correlators in the Regge limit. Higher spin operators have increasingly large contributions, and there is little to be learned from the subleading stress tensor term.

In a holographic CFT, on the other hand, with large-N factorization of correlators and a large gap in the spectrum of higher spin operators, the Regge OPE is under control in the $1/N$ expansion. In this case, from the derivation, it is clear that (2.3.7) includes anything that can be computed by a graviton exchange Witten diagram. In terms of CFT operators, this includes $T_{\mu\nu}$ itself, as well as double-trace operators $[\phi_1\phi_2]$ built from all of the light operators in theory, including products of stress tensors $[TT]$. In a four-point function, it gives the dominant term in the Regge limit whenever the exchange diagram dominates. This breaks down deep into the Regge regime where the exchange of massive higher spin particles (e.g., string states) becomes important.

Single trace contribution

Let's examine the single-trace contribution, which is universal to all CFTs. The Regge OPE (2.3.7) is written in terms of the bulk metric. Using the HKLL prescription, we can rewrite it in terms of boundary CFT operators. Specializing to $D = 4$, the HKLL formula for the metric perturbation is [38]:

$$h_{\mu\nu}(t, y, \vec{x}, z) = \frac{8\pi G_N}{\pi^2 z^2} \int_{t'^2+y'^2+\vec{x}'^2<z^2} dt' dy' d^2\vec{x}' T_{\mu\nu}(t+t', y+iy', \vec{x}+i\vec{x}') + \text{multitrace}\,. \tag{2.3.14}$$

Plugging the single-trace term into (2.3.7) and doing some of the integrals gives the stress tensor contribution to the Regge OPE:[5]

$$\frac{\psi(u,v)\psi(-u,-v)|_T}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \frac{2\lambda_T u}{\pi}\int_{-\infty}^{+\infty} d\tilde{u}\int_{\vec{x}^2\leq -uv} d\vec{x}^2 \frac{-uv-\vec{x}^2}{uv} T_{uu}(\tilde{u},0,i\vec{x})\ , \quad (2.3.16)$$

where $\lambda_T = (2\pi G_N)\Delta_\psi = \frac{10\Delta_\psi}{c_T\pi^2}$.

Although this was derived from gravity, it is completely general, for all CFTs and all values of Δ_ψ. It is a statement about the piece of the stress tensor conformal block that grows in the Regge limit, so it can also be derived directly from CFT. This is most efficiently done using the OPE block formalism [39, 40]. The OPE block for a primary operator O is the complete contribution of O and its descendants to the $\psi\psi$ OPE. Assuming temporarily that x_1 and x_2 are timelike separated, the OPE block for the stress tensor can be computed by integrating $T_{\mu\nu}$ over a codimension-1 spacelike surface [39, 40]

$$\psi(x_1)\psi(x_2)|_T = -\frac{2\lambda_T}{\pi^2}\int_{B(x_1,x_2)} d\Sigma^\mu K^\nu T_{\mu\nu}\ . \quad (2.3.17)$$

Here $B(x_1, x_2)$ denotes a Cauchy surface within the causal diamond $future(x_1)\cap past(x_2)$, which we take to be the ball equidistant from x_1 and x_2. The conformal Killing vector $K^\mu = \frac{-2\pi}{(x_2-x_1)^2}[(x_2-x)^2(x_1-x)^\mu - (x_1-x)^2(x_2-x)^\mu]$ is the generator of time translations within the diamond.

For spacelike separated x_1, x_2, the OPE block is obtained from (2.3.17) by analytic continuation. (Note that this immediately gives the Euclidean OPE block

[5]Derivation: The integral is

$$\frac{\psi(u,v)\psi(-u,-v)|_T}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \frac{2\lambda_T}{\pi^2 v}\int_{t'^2+y'^2+\vec{x}'^2<-uv} dt'dy'd^2\vec{x}'\int_{-\infty}^{\infty} du' T_{uu}\left(\frac{u'}{2}+t', -\frac{u'}{2}+iy', i\vec{x}'\right)\ . \quad (2.3.15)$$

Now shift the contour $u'\to u'-t'+iy'$, and rewrite the integrals over t', y' in the form $\frac{1}{2}\int_{|z|<-uv-t'^2} d^2z T_{uu}(u=u', v=z, \vec{x}=i\vec{x}')$ where $z=t'+iy'$. Finally, do the complex z-integral by assuming the correlator is analytic within the disk of integration: $\int_{|z|\leq R} d^2z f(z) = 2\pi R^2 f(0)$. This assumption is valid in the four-point functions where we will apply (2.3.16).

without 'shadow' contributions.) In our kinematics, with $v < 0 < u$, the full OPE block is

$$\frac{\psi(u,v)\psi(-u,-v)|_T}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \frac{2\lambda_T}{\pi}\int_{-u}^{u} d\tilde{u}\int_{\vec{x}^2\leq -vu\left(1-\frac{\tilde{u}^2}{u^2}\right)} d^2\vec{x}\,\frac{-uv\left(1-\frac{\tilde{u}^2}{u^2}\right)-\vec{x}^2}{v} \tag{2.3.18}$$
$$\times\left[T_{uu}\left(\tilde{u},-\frac{v\tilde{u}}{u},i\vec{x}\right)-2\frac{v}{u}T_{vu}\left(\tilde{u},-\frac{v\tilde{u}}{u},i\vec{x}\right)+\frac{v^2}{u^2}T_{vv}\left(\tilde{u},-\frac{v\tilde{u}}{u},i\vec{x}\right)\right] .$$

In the Regge limit (2.3.1), the ball integral becomes an integral over a slab on the null sheet $v = 0$, and reproduces exactly the gravity result (2.3.16).

In terms of 4-point conformal blocks, (2.3.16) captures precisely the growing part of the block in the Regge limit. For example, in $D = 4$, the stress tensor conformal block has a growing part $\sim \frac{i\bar{z}}{z(z-\bar{z})}$, where $z, \bar{z}$ are the conformal cross ratios. Inserting the operator (2.3.16) into the four-point function reproduces exactly this term, and no more. This requires some care with the contour of integration and is worked out explicitly in appendix A.2.

In the lightcone limit, which is $v \to 0$ with u held fixed, the domain of integration for transverse part of the OPE block is small, and the ball integral becomes a line integral. Thus, we can set $\vec{x} = 0$ in the argument of the stress tensor and (2.3.18) reduces to the lightcone OPE derived in [10],

$$\frac{\psi(u,v)\psi(-u,-v)|_T^{\text{lightcone}}}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \lambda_T v u^2\int_{-u}^{u} d\tilde{u}\left(1-\frac{\tilde{u}^2}{u^2}\right)^2 T_{uu}\left(\tilde{u},0,\vec{0}\right) . \tag{2.3.19}$$

Again this agrees with the length operator (2.3.6) upon using HKLL to replace $h_{uu} \to T_{uu}$.

The analysis is similar in general dimensions, leading to the Regge OPE

$$\frac{\psi(u,v)\psi(-u,-v)|_T}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \frac{\Delta_\psi 2^d \pi^{\frac{1}{2}-d}\Gamma(\frac{d+2}{2})\Gamma(\frac{d+3}{2})}{c_T(d-1)} u \int_{-\infty}^{+\infty} d\tilde{u} \int_{\vec{x}^2\leq -uv} d^{d-2}\vec{x}\, \frac{-uv-\vec{x}^2}{uv} T_{uu}\left(\tilde{u},0,i\vec{x}\right) \ . \tag{2.3.20}$$

As noted above, this is just the single-trace contribution to the Regge OPE (2.1.2) from the operator $T_{\mu\nu}$ and its derivatives. We will see below that this term has a direct interpretation in terms of a time delay in the shockwave geometry.

Note that even for holographic CFTs, the above expression is not equivalent to the full Regge OPE (2.1.2). The term in (2.3.20) is just the stress tensor contribution to the Regge OPE, which can also receive significant contributions from all double-trace operators $[\phi_1\phi_2]$ built from any light operators in theory, including products of stress tensors $[TT]$. However, as we will show later, in certain four-point functions of smeared operators, the OPE (2.3.20) gives the dominant contribution in the Regge limit.

Regge OPE for other operators

Although we will not use it anywhere in this chapter, the same analysis can be applied to other operators. For the production of an operator X with spin ℓ and dimension Δ, the full OPE block can be written as an integral over the causal diamond between x_1 and x_2 [39, 40], and analytically continued to spacelike separation. In appendix A.3 we take the Regge limit and find

$$\frac{\psi(u,v)\psi(-u,-v)|_X}{\langle\psi(u,v)\psi(-u,-v)\rangle} = \pi^{\frac{1-d}{2}}2^{\Delta}\frac{\Gamma(\frac{\Delta+\ell+1}{2})}{\Gamma(\frac{\Delta+\ell}{2})}\frac{\Gamma(\Delta-d/2+1)}{\Gamma(\Delta-d+2)}\frac{C_{\psi\psi X}}{C_X}$$
$$\times\frac{(-uv)^{\frac{d-\ell-\Delta}{2}}}{u^{1-\ell}}\int_{-\infty}^{+\infty}d\tilde{u}\int_{\vec{x}^2\leq -uv}d^{d-2}\vec{x}(-uv-\vec{x}^2)^{\Delta-d+1}X_{uu\cdots u}(\tilde{u},0,i\vec{x}) \tag{2.3.21}$$

This reproduces the growing part of the 4-point conformal block.

2.4 Causality constraints revisited

2.4.1 The Engelhardt-Fischetti criterion

Causality is subtle in a theory with gravity, because local light cones can always be 'opened' by a diffeomorphism. At the least, causality should be respected at infinity, in the sense that a probe sent from infinity that passes through some geometry should return to infinity no faster than it would in vacuum [44, 53, 54]. That is, all time delays should be non-negative. In a geometry that is asymptotically AdS, the statement is that lightcones on the conformal boundary must be respected by the bulk. In AdS/CFT, the same criterion is imposed by causality of the dual CFT [55].

However, it remains an open question exactly what feature of the bulk theory ensures boundary causality. Gao and Wald showed that in Einstein gravity, the averaged null energy condition for bulk matter implies boundary causality [44]. More generally, it is respected in any theory of gravity obeying the averaged null curvature condition, $\int duR_{uu} \geq 0$. But is this condition also necessary? Perhaps not — there are geometries which violate the curvature condition, but preserve boundary causality [43]. It is not known whether such solutions can be supported by physical matter. Geometrically, the necessary and sufficient condition for boundary causality, at the perturbative level, was derived by Engelhardt and Fischetti [43]. It is simply the condition

$$\int_\gamma d\lambda h_{\lambda\lambda} \geq 0 \ , \tag{2.4.1}$$

where the integral is over a complete null geodesic γ in AdS. The integral computes a time delay, so the same condition has been imposed in specific states in prior work (for example [5, 13, 21, 56–58]). Any such null geodesic can be viewed as a constant-z path in some Poincare patch.

The quantity in (2.4.1) is precisely the operator that appears in the Regge OPE (2.3.7). And, the chaos bound [41] implies that this contribution to the correlator, including the overall minus sign, must be negative. Therefore, the chaos bound, together with the Regge OPE, constitutes a derivation of (2.4.1), now viewed as an expectation value of this operator in the quantum theory. (See [9] for more discussion on the connection between chaos and the Regge limit of the 4-point function.) This is the Regge analogue of the averaged null energy condition (ANEC) as derived from causality in [10].

Unlike the ANEC, the conclusion here comes with the caveat that we have only shown (2.4.1) is positive in a certain class of states. Exactly which states depends on the assumptions, and how we interpret the formula:

(i). If h_{uu} is interpreted as the full metric perturbation operator, with contributions from multitrace operators as well as the stress tensor, then the conclusion is that the chaos bound implies (2.4.1) in any holographic CFT dual gravity, in any state perturbatively close to the vacuum. By 'gravity', we mean that the bulk theory has a large gap without any single trace higher spin states, but may have large higher curvature corrections. The state does not need to be geometric; for example, it could be a superposition of classical geometries, with h_{uu} evaluated as an expectation value. In general, the multitrace contributions are important, so in CFT language, this inequality would be quite complicated, and cannot be written in terms of just the operator $T_{\mu\nu}$.

(ii). In certain states, single-trace exchange dominates, and we can replace $h_{uu} \to h_{uu}|_T$, the single-trace, stress-tensor contribution to HKLL. These are states $|\phi\rangle$ in which $\langle\phi|\psi\psi|\phi\rangle$ is dominated by the stress tensor conformal block. In a large-N CFT with a large gap in the higher spin spectrum, this includes states created by a heavy (but not backreacting) operator insertion $\phi(x)$, with $c_T \gg \Delta_\phi \gg 1$, as well as superpositions of such states, and states with multiple heavy operator insertions. We will show below that it also includes certain states produced by smeared light operators.

In gravity language, we have shown that the geodesic time delay is positive in any state near the vacuum.

So far, this argument is not enough to reproduce the "$a = c$" constraints derived from graviton causality in [5]. We will derive these below, by projecting out double trace contributions to correlators of light operators. In the rest of this section, we focus on the case of 4 heavy scalars, and describe how the Regge OPE is directly related to a bulk time delay.

2.4.2 Time delays for probe particles

To make this more concrete, we will now compare two different calculations of a 4-point correlator: The CFT calculation using the Regge OPE, and the bulk calculation using geodesic lengths in the shockwave background. The point is that Δv, the time delay as a scalar particle crosses a shock, is computed precisely by the Regge OPE for the stress tensor alone, (2.3.16). This is almost clear from the gravity formula (2.3.7) but there are some subtleties in the prescription for how to apply this OPE, especially as the probe geodesic moves past the shockwave source into the bulk.

Setup

We consider the scalar four-point function

$$G \equiv \frac{\langle\psi(x_1)\phi(x_3)\phi(x_4)\psi(x_2)\rangle}{\langle\psi(x_1)\psi(x_2)\rangle\langle\phi(x_3)\phi(x_4)\rangle} \ . \tag{2.4.2}$$

There are two cases where the stress tensor term (2.3.16) is the dominant contribution, without any contamination from double trace or other operators. One is the lightcone limit, in any CFT, where it dominates because $T_{\mu\nu}$ is the operator of lowest twist. The other is the Regge limit in a large-N, holographic CFT, dual to gravity (possibly with higher curvature corrections but without higher spin fields), with all four operators heavy: $c_T \gg \Delta_\psi, \Delta_\phi \gg 1$.

We'll start with the Regge limit. In this case the OPE is controlled by $1/N$, and double trace contributions from $[\psi\psi]$ and $[\phi\phi]$ are suppressed by their large dimensions. In order to utilize the shockwave dictionary (2.1.1), with a real shockwave geometry, we choose the imaginary kinematics

$$x_1 = -x_2 \equiv (u_\psi, v_\psi, \vec{x}_\psi) \equiv (i\delta,\ \frac{iz_0^2}{\delta},\ \vec{0})\ . \tag{2.4.3}$$

The points x_1, x_2 are conjugate, so G is an expectation value:[6]

$$G = \frac{\langle\psi_\infty|\phi(x_3)\phi(x_4)|\psi_\infty\rangle}{\langle\psi_\infty|\psi_\infty\rangle\langle\phi(x_3)\phi(x_4)\rangle}\ . \tag{2.4.4}$$

The shockwave lies on the Rindler horizon, at $u = 0$. We also insert ϕ symmetrically across the Rindler wedge:

$$x_3 = (u_\phi, v_\phi, \vec{x}_\phi)\ , \quad x_4 = (-u_\phi, -v_\phi, \vec{x}_\phi)\ , \quad \text{with} \quad v_\phi < 0 < u_\phi\ . \tag{2.4.5}$$

The limit $\delta \to 0$ is taken with $u_\phi, v_\phi, \vec{x}_\phi, z_0$ fixed and $O(1)$.

This setup is illustrated in figure 2.6.

[6]Note that these are not the kinematics used in the derivation of the chaos bound. The sign constraints discussed in section 2.4 are derived using kinematics where x_1 and x_2 are related by a reflection across the Rindler horizon, rather than a Hermitian conjugation [41] (see also [9]). However, the chaos bound does imply — indirectly — that Δv in the current calculation must be positive, because it is computed by the same operator, which must have positive expectation value in any state. In order to make a more direct connection between the chaos argument and the bulk time delay would require an imaginary metric in the bulk.

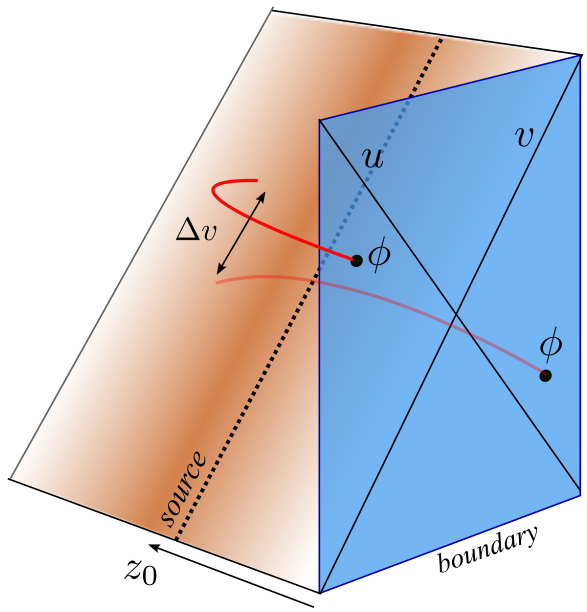


Figure 2.6: Setup for the four-point function. The metric is a real shockwave, created by ψ inserted at imaginary kinematics. ϕ is a probe, and $\langle\psi|\phi\phi|\psi\rangle$ is computed from the geodesic length. The geodesic jumps by Δv across the shockwave. The transverse directions $\vec{x}$ also play a role but are suppressed in the figure.

Geodesics in the shockwave

On the gravity side, $|\psi_\infty\rangle$ is the planar shockwave with metric (2.2.1) and $E = \Delta_\psi/(2\delta)$, so G is the two-point function $\langle\phi\phi\rangle_{shock}$ in this background. Since ϕ is heavy, it is computed by a geodesic length:

$$G = \frac{\langle\phi(x_3)\phi(x_4)\rangle_{shock}}{\langle\phi(x_3)\phi(x_4)\rangle_{vacuum}} = \exp\left[-\Delta_\phi(L_{shock}(x_3,x_4) - L_{vac}(x_3,x_4))\right] \ . \tag{2.4.6}$$

The metric is empty AdS away from the shock. When the geodesic crosses the shockwave at $u = 0$, heading in the positive-u direction, it jumps by $v \to v + \Delta v$, with

$$\Delta v(z_c, \vec{x}_c) = z^2 h\big|_{z=z_c,\vec{x}=\vec{x}_c} \tag{2.4.7}$$

where $(z_c, \vec{x}_c)$ are coordinates at the crossing point, and h is given by h_{uu}^{Shock} in (2.2.1) with the delta function stripped off, *i.e.*, $h = \int_{0^-}^{0^+} du h_{uu}$. In principle, finding the full geodesic is just a matter of patching together empty-AdS geodesics with the correct endpoints. This is somewhat complicated because crossing the shockwave also imparts transverse $\vec{x}$-momentum to the probe, but we only need the first perturbative $O(E)$ term, which is straightforward. The calculation is exactly the same as that of section (2.3.1), yielding

$$G \approx 1 - \frac{\Delta_\phi u_\phi}{2} \int_{-u_\phi}^{u_\phi} du' \left(1 - \frac{u'^2}{u_\phi^2}\right) h_{uu}^{Shock} = 1 - \frac{\Delta_\phi u_\phi}{2} \frac{\Delta v(z_c, \vec{x}_c)}{z_c^2} \tag{2.4.8}$$

with the crossing point at

$$z_c = \sqrt{-u_\phi v_\phi}, \quad \vec{x}_c = \vec{x}_\phi \ . \tag{2.4.9}$$

For example, in $D = 4$ with $\vec{x}_\phi = 0$, the time delay (2.4.7) in the metric (2.2.1) is

$$\Delta v = 8EG_N \times \begin{cases} \frac{z_c^4}{z_0^2 - z_c^2} & z_c < z_0 \\ \frac{z_0^4}{z_c^2 - z_0^2} & z_c > z_0 \end{cases} . \tag{2.4.10}$$

Using $E = \Delta_\psi/(2\delta)$, $c_T = \frac{5}{\pi^3 G_N}$, and (2.4.3), this becomes

$$G - 1 \approx -\frac{10\Delta_\phi\Delta_\psi}{\pi^3 c_T |v_\phi u_\psi|} \times \begin{cases} \frac{|u_\phi v_\phi|^2}{|u_\psi v_\psi| - |u_\phi v_\phi|} & |u_\psi v_\psi| > |u_\phi v_\phi| \\ \frac{|u_\psi v_\psi|^2}{|u_\phi v_\phi| - |u_\psi v_\psi|} & |u_\psi v_\psi| < |u_\phi v_\phi| \end{cases} \tag{2.4.11}$$

Here we can see the symmetry between the roles of the shockwave source ψ and probe field ϕ, as evident from (2.4.2). It does not matter which pair is taken into the shockwave limit, since the two options are conformally equivalent. The non-analyticity at $u_\psi v_\psi = u_\phi v_\phi$ in (2.4.11) is where the probe geodesic hits the shockwave source, and is smoothed out by any nonzero transverse separation $\vec{x}_\phi - \vec{x}_\psi$.

The general answer, at non-zero $\vec{x}_\phi$, can be written using the conformal cross ratios $z, \bar{z}$. In the Regge limit,[7]

$$\begin{aligned} z\bar{z} &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \approx \frac{16 u_\psi v_\phi}{u_\phi v_\psi} & (2.4.12) \\ z + \bar{z} &\approx 1 - \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = 4\frac{\vec{x}_\phi^2 + z_0^2 + z_c^2}{u_\phi v_\psi} & (2.4.13) \end{aligned}$$

The variable ρ that appears in h_{uu}^{shock}, evaluated at the crossing point, is

$$\rho_c = \sqrt{\frac{(z_c - z_0)^2 + \vec{x}_\phi^2}{(z_c + z_0)^2 + \vec{x}_\phi^2}} = \frac{1 - \sqrt{\bar{z}/z}}{1 + \sqrt{\bar{z}/z}} \ . \tag{2.4.14}$$

Therefore, in $D = 4$,

$$G \approx 1 + \frac{40 \Delta_\phi \Delta_\psi}{\pi^3 c_T} \frac{i\bar{z}}{z(z - \bar{z})} \tag{2.4.15}$$

Note that this single expression (which we will see is equal to a Regge conformal block) accounts for both of the piecewise answers in (2.4.11). The jump comes from the fact that in the limit $\vec{x}_\phi \to 0$, the cross ratios (defined to be smooth at finite $\vec{x}_\phi$) become non-analytic in the coordinates: $z/\bar{z} = \max(z_0^2/z_c^2, z_c^2/z_0^2)$.

We can also make a direct connection between the correlator and the bulk equations of motion. Since the correction to the correlator was just the bulk metric h_{uu}^{Shock} with a delta function stripped off, it obeys the Einstein equation, integrated over u:[8]

$$\left(\partial_{z_c}^2 - 3\frac{\partial_{z_c}}{z_c} + \partial_{\vec{x}_\phi}^2\right)\left(\frac{\pi^2 c_T |v_\phi u_\psi|}{20\Delta_\phi \Delta_\psi}\right)(G - 1) = z_0^3 \delta(z_c - z_0)\delta(\vec{x}_\phi - \vec{x}_\psi) \ . \tag{2.4.16}$$

[7]Both cross ratios are of the form $z, \bar{z} \sim -i \times (\textit{positive})$, and we choose (arbitrarily) the solution of the quadratic equation such that $\bar{z}/z \to 0$ in the lightcone limit $v_\phi \to 0^-$. This implies $\bar{z}/z$ is real and positive for any choice of external points within the regime considered here.

[8]The differential operator on the left is the conformal Casimir operator in the Regge limit, $z^2\partial^2 + \bar{z}^2\partial^2 + \frac{2z\bar{z}}{z-\bar{z}}(\partial - \bar{\partial})$. Interestingly, acting on the block it produces a delta function source in Lorentzian signature. It may be fruitful to understand the role of these delta functions in the conformal bootstrap.

Although we have assumed Einstein gravity at intermediate steps, the final answer (2.4.15) written in terms of c_T is also valid in the presence of higher derivative corrections (both E and G_N get rescaled, but EG_N is unchanged).

CFT calculation using the Regge OPE

Let's reproduce this from CFT. One way to proceed is to use the Dolan-Osborn expression for the stress tensor conformal block; upon analytic continuation to the Regge regime, this gives directly (2.4.15), as shown in appendix A.2. Instead we will use the Regge OPE (2.3.7) to illustrate how to apply it. Since ψ is the operator that is inserted in the shockwave limit, the direct approach would be to use (2.3.7) in the form

$$\psi\psi \sim 1 - \frac{\Delta_\psi v_\psi}{2} \int_{-\infty}^{\infty} dv h_{vv}|_T, \tag{2.4.17}$$

and then to calculate $\langle \phi \int dv h_{vv} \phi \rangle$. The notation $|_T$ indicates that only the single-trace stress tensor part of $h_{\mu\nu}$ is included (via HKLL as discussed in section 2.3.2). This will be useful below, but first, for comparison to the gravity result (2.4.8) it is more informative to contract the probes $\phi\phi$ rather than the sources $\psi\psi$. That is, we use

$$\phi\phi \sim 1 - \frac{\Delta_\phi u_\phi}{2} \int du h_{uu}|_T \ . \tag{2.4.18}$$

Despite that fact that we have not taken $v_\phi, 1/u_\phi \to 0$, this OPE can be used because in the conformal cross-ratio it does not matter which operator is the source, and which is the probe. (In other words, we could boost so that $\phi\phi$ are Regge-separated with $\psi\psi$ near the origin, apply the Regge OPE to $\phi\phi$, then boost back.)

Formally, plugging (2.4.18) into the state $|\psi_\infty\rangle$ gives a correlator manifestly equal to the gravity result (2.4.8), since $\Delta v = \int duz^2 h_{uu}$. But to confirm this by direct evaluation in the CFT, we need to account for some subtleties involving the choice of u-contour. Converting (2.4.18) to a CFT expression using (2.3.16), then evaluating in the shockwave state (with $\vec{x}_\phi = 0$, $D = 4$) we find:

$$\begin{aligned} G - 1 &= -\frac{20 u_\phi \Delta_\phi}{\pi^3 c_T} \int du \int_{\vec{x}^2 < |u_\phi v_\phi|} d^2\vec{x} \frac{|u_\phi v_\phi| - \vec{x}^2}{|u_\phi v_\phi|} \langle \psi_\infty | T_{uu}(u, v = 0, i\vec{x}) | \psi_\infty \rangle / \langle \psi_\infty | \psi_\infty \rangle \\ &= \frac{320 \delta^4 \Delta_\phi \Delta_\psi z_0^6}{3\pi^4 c_T |v_\phi|} \int du \int_0^{z_c} dr \frac{r(r^2 - z_0^2)^2 (r^2 - z_c^2)}{(u^2 z_0^4 + (r^2 - z_0^2)^2 \delta^2)^3} \end{aligned} \tag{2.4.19}$$

where on the second line we plugged in the conformal three-point function (3.4.4), and $z_0 = |v_\psi u_\psi|^{1/2}, z_c = |v_\phi u_\phi|^{1/2}$. For $z_c < z_0$, the integrals are straightforward, with the u integral done first. The correct prescription for the u integral can be derived by boosting the ϕ's, integrating $\int_{-\infty}^{\infty} du$, then boosting back; the upshot is that the integral just picks up a residue. After doing the r-integral, the final result is equal to the first line of (2.4.11). Therefore, we have exactly reproduced the case where the probe geodesic crosses the shockwave closer to the boundary than the source particle.

In the other case, where the geodesic probe crosses the shockwave past the source, *i.e.,* $z_c > z_0$, the integral in (2.4.19) diverges. The easiest way around this obstacle is to swap the role of source and probe, *i.e.,* to use (2.4.17) instead. Clearly this reproduces the gravity result on the second line of (2.4.11), since it differs from the first line just by swapping the two operators.

Therefore, we have reproduced the full gravity answer using the Regge OPE.

Lightcone limit

Everything in sections 2.4.2 - 2.4.2 can also be applied in the lightcone limit, where $u \to 0$ with v held fixed. In this case, there is perfect agreement in all CFTs, regardless of whether they have a holographic dual, since the entire calculation happens near the boundary of AdS. On the CFT side, the perturbative analysis is valid because instead of $1/N$, the corrections to the correlator are suppressed by positive powers of u. On the bulk side, the perturbative geodesic calculation is valid because now $z_c \ll z_0$, and in that limit, the discontinuity in the geodesic where it hits the shockwave is also suppressed by u. We omit the details but the result in the end is just the lightcone limit of (2.4.11), in both boundary and CFT. This result demonstrates how log terms in the lightcone conformal block are related to time delays [10], and provides a precise equivalence between two different derivations of the averaged null energy condition: The holographic derivation [45] using geodesic lengths, and the causality derivation use the lightcone OPE [10].

2.5 Shockwaves from light operators

So far, all of our discussion of shockwaves has required the insertion of two heavy operators ψ, with $\Delta_\psi \gg 1$. From a bulk point of view, heavy operators are massive probe particles that travel on geodesics. Now we turn to another way to make shockwaves, using wavepackets of light operators, designed to travel on geodesics in the bulk. We mostly restrict to $D = 4$ for simplicity, but the results should generalized straightforwardly to any $D > 2$.

2.5.1 Smeared operators

The smearing procedure that we will use to produce shockwaves is identical, up to a conformal transformation, to that used in [9]. There, it was motivated by a sequence of complexified rotations, shifts, and limits from the kinematics of Hofman and Maldacena [13]. Here we work in a different conformal frame where the smearing is simpler to understand using a null inversion.

The starting point is an operator inserted at some fixed, real Lorentzian time $t = t_0 > 0$, smeared over complex positions with a Gaussian profile:

$$W'(t_0) = \int dy d^{d-2}\vec{w} e^{-(y^2+\vec{w}^2)/D^2} \phi(t_0, iy, i\vec{w}) \ . \tag{2.5.1}$$

We implicitly cutoff the Gaussian outside some window, so the smearing has finite support; details off the cutoff won't matter, and the dominant contributions to the integrals will always come from a region where the Gaussian factor can be ignored. One can also think of this operator as a momentum space insertion, Wick rotated in all directions:

$$\int d\tau d^{d-1}x \, e^{-(\tau^2+x^2)/D^2} e^{-i\omega\tau} \phi(t = i\tau, ix) \ . \tag{2.5.2}$$

This is not quite the same operator, but is equal inside the correlation functions we consider if evaluated at the saddlepoint $t = t_0 \sim \omega D^2$.

Now perform the null inversion (2.2.21). The smeared operator (with the arguments of ϕ written in null coordinates u, v) becomes

$$W(t_0) = \int dy d\vec{w} e^{-(y^2+\vec{w}^2)/D^2} \left(\frac{t_0 + iy}{z_0}\right)^{-\Delta_\phi} \phi(u, v, \vec{x}) \tag{2.5.3}$$

where under the integral, the coordinates are parameterized as

$$u = t_0 - iy + \frac{\vec{w}^2}{t_0 + iy} \ , \qquad v = -\frac{z_0^2}{t_0 + iy} \ , \qquad \vec{x} = i\frac{z_0\vec{w}}{t_0 + iy} \ . \tag{2.5.4}$$

The center of the smeared operator, $y = \vec{w} = 0$, is

$$u_{center} = t_0, \qquad v_{center} = -\frac{z_0^2}{t_0} \ . \tag{2.5.5}$$

Therefore, the Regge limit is $t_0 \to 0$. The limit is taken with D/t_0 fixed but large, so that all of the coordinates $(t_0, y, \vec{w})$ effectively scale together towards zero.

We will insert a second wavepacket which is obtained from W by sending $t_0 \to -t_0$. Importantly, after the null inversion, this is equivalent to reflecting across the Rindler horizon, up to a fixed phase. Rindler reflection sends

$$(u, v, \vec{x}) \to (\bar{u}, \bar{v}, \overline{\vec{x}}) \equiv (-u^*, -v^*, \vec{x}^*) \ . \tag{2.5.6}$$

Thus we define

$$\overline{W}(t_0) \equiv (-1)^{\Delta_\phi} W(-t_0) = \int dy d\vec{w} e^{-(y^2+\vec{w}^2)/D^2} \left(\frac{t_0 - iy}{z_0}\right)^{-\Delta_\phi} \phi(-u^*, -v^*, \vec{x}^*) \ , \tag{2.5.7}$$

where $u, v, \vec{x}$ are given by (2.5.4).The phase comes from the conformal factor in the null inversion.

Rindler reflection is important for positivity/causality constraints, because Rindler-symmetric correlators have properties akin to reflection positivity in Euclidean signature; see [10, 41, 59] for a detailed discussion.

Positivity at timelike separation

As an aside, note that under the null inversion, the fact that Rindler-symmetric correlators are positive leads to a positivity condition for timelike separated insertions. The simplest example of this positivity condition is the vacuum two-point

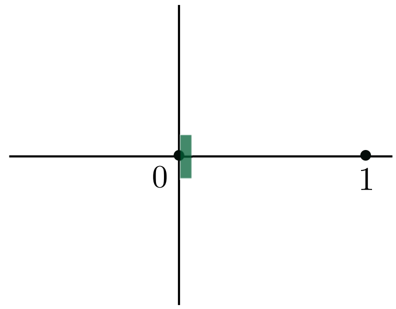


Figure 2.7: The smearing region on the z and $\bar{z}$ planes is the shaded rectangle near the origin. The Regge limit is $z, \bar{z} \to 0$.

function, $\langle O(x_1)O(x_2)\rangle = (x_{12})^{-2\Delta}$. With real t and any x, this obviously obeys the positivity condition

$$(-1)^{\Delta}\langle O(t,x)O(-t,x)\rangle = |2t|^{2\Delta} > 0 \ . \tag{2.5.8}$$

Rindler positivity, after a null inversion, implies that a similar relation holds for n-point functions, with a specific phase dictated by the conformal weights. This is somewhat surprising — it is a positivity condition in unitary CFTs that applies to timelike separate insertions. In what follows, we will work in the coordinates where operators are inserted symmetrically across the Rindler horizon, so chaos/causality bounds are a consequence of Rindler reflection positivity. Alternatively, one can work directly in the $(t, y, \vec{w})$ patch, with the shockwave inserted near the origin; in this approach the chaos/causality bounds follow from a timelike positivity condition like (2.5.8).

Cross ratios

Consider these wavepackets inserted in a 4-point function,

$$G = \langle \overline{W}(t_0)W(t_0)\phi'(x_3)\phi'(x_4)\rangle \ . \tag{2.5.9}$$

This is a double integral of $\langle\phi(x_1)\phi(x_2)\phi'(x_3)\phi'(x_4)\rangle$, where

$$x_1 = (u_1, v_1, \vec{x}_1) \tag{2.5.10}$$

$$x_2 = (-u_2^*, -v_2^*, \vec{x}_2^*) \tag{2.5.11}$$

with $u_i, v_i, \vec{x}_i$ defined as in (2.5.4). Let's choose the other operators to also be symmetric across the Rindler horizon,

$$x_3 = -x_4 = (u = -1, v = 1, \vec{0}) \ . \tag{2.5.12}$$

The conformal cross-ratios for this situation, in the Regge limit where $t_i, y_i, \vec{w}_i$ all scale toward zero, are

$$z\bar{z} \approx \frac{4}{z_0^2}(4t_0^2 + (\vec{w}_1 - \vec{w}_2)^2 + (y_1 - y_2)^2) \tag{2.5.13}$$

$$z + \bar{z} \approx 4t_0(1 + \frac{1}{z_0^2}) - 2i(y_1 - y_2)(1 - \frac{1}{z_0^2}) \ . \tag{2.5.14}$$

On the z and $\bar{z}$ complex planes, the range of the smearing integrals lies within the shaded region of figure 2.7. The cross ratios always stay in the right half-plane, Re $z, \bar{z} > 0$. This is an important feature of this smearing procedure, required in order to apply the chaos bound to derive causality constraints. It differs from the more natural-looking, real-momentum wavepackets used in [25, 26], which involve smearing into the bulk-point regime with Re $z, \bar{z} < 0$.

2.5.2 Shockwaves in the smeared OPE

We assume ϕ is a light operator, with dimension $O(1)$. We will show that the smeared operators W behave similarly to heavy operators, in the following sense: First, the stress tensor in the state $W(t_0)|0\rangle$ is exactly that of the shockwave

(2.2.5), localized on a null plane, with energy

$$E = \frac{i\Delta_\phi}{2t_0}\left(1 - \frac{d-1}{2\Delta_\phi}\right) . \tag{2.5.15}$$

(The shockwave is real for imaginary t_0). Second, the leading contribution to the Regge OPE is

$$\frac{\overline{W}(t_0)W(t_0)}{\langle\overline{W}(t_0)W(t_0)\rangle} \approx 1 - iEz_0^2\int_{-\infty}^{\infty} dv h_{vv}(u=0, v, z=z_0, \vec{x}=0) . \tag{2.5.16}$$

On the right-hand side is the shockwave operator, just as in the heavy-scalar Regge OPE, (2.3.7). This means that the smearing effectively projects out double-trace $[\phi\phi]$ operators, which would contribute at the same order if we just inserted local operators without smearing. That is, the correlator $\langle\overline{W}W\phi'\phi'\rangle$, in the Regge limit, has contributions from T and $[\phi'\phi']$, but no contributions from $[\phi\phi]$ at leading order.

We will derive (2.5.16) in $D = 4$, from both gravity and CFT. The formula for energy in general D (2.5.15) was obtained by computing $\langle\overline{W}\int duT_{uu}W\rangle$ and comparing to the prediction of (2.5.16), and matches the energy in the OPE that we will find for $D = 4$.

2.5.3 Gravity derivation

On the gravity side, (2.5.16) is derived by smearing the vertex Witten diagram in (2.3.8). The smearing integrals depend only on the relative separations up to an overall factor, so we only need to smear one of the operators. The Gaussian factors also drop out since the integral is dominated near the origin of the $(t_0, y, \vec{w})$

coordinates. Let

$$x_1 = (-u_{center}, -v_{center}, \vec{0}), \qquad x_2 = (u, v, \vec{x}) \tag{2.5.17}$$

with the coordinates given by (2.5.4). Performing the integrals, we find

$$\frac{\int dyd\vec{w}(t_0 + iy)^{-\Delta_\phi} \Pi_{\alpha\beta}(x_1, x_2; z', x')}{\int dyd\vec{w}(t_0 + iy)^{-\Delta_\phi} \langle \phi(x_1)\phi(x_2)\rangle} = \frac{1}{2} h^{Shock}_{uu} \delta^u_\alpha \delta^u_\beta \tag{2.5.18}$$

where h^{Shock}_{uu} is the shockwave metric (2.2.1), with E given by (2.5.15).

Stripping off the gravity propagator as we did in sections 2.3.1-2.3.1, this is equivalent to the OPE formula (2.5.16).

2.5.4 CFT derivation from conformal Regge theory

Now we will derive the same OPE formula (2.5.16) in a large-N, holographic CFT, using conformal Regge theory [32].[9] As usual, we assume a large gap in the spectrum of single trace higher spin primaries, so that the only contribution to the Regge amplitude comes from the graviton. Conformal Regge theory provides a way to compute 4-point functions that automatically includes double-trace as well as single-trace exchanges. In terms of 4-point correlators, the statement of (2.5.16) is that the smeared 4-point function can be computed as follows: (i) replace the wavepackets by local operators at the centers, and (ii) drop double-trace $[\phi\phi]$ contributions. We will demonstrate this by smearing the conformal Regge amplitude.

The derivation follows [32] and is very similar to [25, 26] so we will be brief. The starting point in conformal Regge theory is the 4-point correlator written as

[9]We thank S. Caron-Huot and A. Zhiboedov for discussions of conformal Regge theory and this calculation.

a conformal partial-wave expansion

$$G(z,\bar{z}) \equiv \frac{\langle\phi\phi\psi\psi\rangle}{\langle\phi\phi\rangle\langle\psi\psi\rangle} = \sum_{\Delta,J} C_{\phi\phi\mathcal{O}} C_{\psi\psi\mathcal{O}} g_{\Delta,J}(z,\bar{z})$$
$$= \sum_{J} \int d\nu \; b_J(\nu^2) F_{\nu,J}(z,\bar{z}), \tag{2.5.19}$$

where conformal partial waves $F_{\nu,J}$ are related to the conformal blocks by

$$F_{\nu,J}(z,\bar{z}) = \kappa_{\nu,J} g_{h+i\nu,J}(z,\bar{z}) + \kappa_{-\nu,J} g_{h-i\nu,J}(z,\bar{z}), \tag{2.5.20}$$

with $h \equiv \frac{d}{2}$ and

$$\kappa_{\nu,J} = \frac{i\nu}{2\pi K_{h+i\nu,J}} \tag{2.5.21}$$

where $K_{h+i\nu,J}$ is given explicitly in [32]. We can evaluate the second line of (2.5.19) by closing the contour in ν below the real axis since the partial waves vanish exponentially as $\nu \to -i\infty$. In order for the contour integral to evaluate to the conformal block expansion $b_J(\nu^2)$ must have poles

$$b_J(\nu^2) \sim \frac{C_{\phi\phi\mathcal{O}} C_{\psi\psi\mathcal{O}} K_{\Delta,J}}{\nu^2 + (\Delta-h)^2} \quad \text{as} \quad \nu^2 \to (\Delta-h)^2. \tag{2.5.22}$$

Note that since we are interested in the Regge limit of the correlator we will be concerned with the analytic continuation of the partial waves to the second sheet, taking z around 1 with $\bar{z}$ held fixed.

The next step is to use the Sommerfeld-Watson transform to analytically continue this expansion away from integer spins by replacing the sum over spins into a contour integral

$$G = \int_{\mathcal{C}} \frac{dJ}{2\pi i} \frac{\pi}{\sin(\pi J)} \int d\nu \; b_J(\nu^2) \left(F_{\nu,J}(z,\bar{z}) + F_{\nu,J}\left(\frac{z}{z-1}, \frac{\bar{z}}{\bar{z}-1} \right) \right), \tag{2.5.23}$$

where $\mathcal{C}$ is the contour enclosing the poles of the denominator corresponding to the integer spins.[10] At large N, deforming this contour to enclose the Regge pole (2.5.22) and separating out the identity contribution one obtains [32]

$$G = 1 + \int d\nu \; a(\nu)(z\bar{z})^{\frac{1-j(\nu)}{2}} \Omega_{i\nu}(\bar{z}/z), \tag{2.5.24}$$

where

$$\begin{aligned} a(\nu) =& \frac{\beta(\nu)\gamma(\nu)\gamma(-\nu)\left(\pi^{h-1}2^{j(\nu)-1}e^{-\frac{i}{2}\pi j(\nu)}\right)}{\sin\left(\frac{\pi j(\nu)}{2}\right)}, \\ \beta(\nu) =& \frac{\pi j'(\nu)\mathrm{K}_{\Delta(j(\nu)),j(\nu)}}{4\nu} C_{\phi\phi j(\nu)} C_{\psi\psi j(\nu)}, \\ \gamma(\nu) =& \Gamma\left(\frac{2\Delta_\phi - h + i\nu + j(\nu)}{2}\right)\Gamma\left(\frac{2\Delta_\psi - h + i\nu + j(\nu)}{2}\right) \\ \Omega_{i\nu}(x) =& \frac{i\nu x^{\frac{1}{2}-\frac{i\nu}{2}}}{\pi^2(x-1)} - \frac{i\nu x^{\frac{1}{2}+\frac{i\nu}{2}}}{\pi^2(x-1)}. \end{aligned} \tag{2.5.25}$$

Assuming that the higher spin single-trace primary operators have scaling dimensions larger than Δ_{gap} and that the stress tensor is the lowest dimension spin-2 operator gives us an ansatz for the leading Regge trajectory

$$j(\nu) = 2 + \frac{1}{\Delta_{\text{gap}}^2}\left(\nu^2 + \frac{\Delta_T^2}{4}\right) + \mathcal{O}(\Delta_{\text{gap}}^{-4}). \tag{2.5.26}$$

Using this ansatz we find

$$G = 1 + C_{\phi\phi T}C_{\psi\psi T}\int d\nu \frac{90\pi\Gamma\left(\Delta_\psi - \frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+\Delta_\psi\right)\Gamma\left(\Delta_\phi - \frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+\Delta_\phi\right)}{(\nu^2+4)\,\Gamma\left(\Delta_\psi - 1\right)\Gamma\left(\Delta_\psi + 1\right)\Gamma\left(\Delta_\phi - 1\right)\Gamma\left(\Delta_\phi + 1\right)} \frac{\Omega_{i\nu}(\bar{z}/z)}{\sqrt{z\bar{z}}}. \tag{2.5.27}$$

We can evaluate this expression by evaluating the contour integral for each term in Ω separately by closing the contour in the upper or the lower half plane accordingly. The two contributions are identical since Ω is symmetric under $\nu \to -\nu$. Therefore,

[10] There is a subtelty in deforming the contour which requires separating the even and odd spin contributions. However, we are only concerned with even spin contributions since we are considering the OPE of identical scalars.

we will consider only one of the terms. Note that in the upper half plane the integrand has a pole at $\nu = 2i$ corresponding to the stress-tensor contrubtion as well as poles at $\nu = 2i(n + \Delta_\phi)$ and $\nu = 2i(n + \Delta_\psi)$ for all positive integers n, corresponding to the exchange of the double trace operators.

We are now ready to apply the smearing procedure described in section 2.5.1 to the correlator. To this end, we start by smearing the partial-wave[11]

$$\begin{aligned}
\tilde{\Omega}_{i\nu}(t_0, z_0) &\equiv \int dyd\vec{w}\frac{1}{\left(\frac{(t_0+iy_1)(t_0-iy_2)}{z_0^2}x_{12}^2\right)^{\Delta_\phi}}\frac{\Omega_{i\nu}(\bar{z}/z)}{\sqrt{z\bar{z}}} \\
&= \int dyrdr\left(4t_0^2 + r^2 + y^2\right)^{-\Delta_\phi}\left(\frac{i\nu z^{-\frac{i\nu}{2}}\bar{z}^{\frac{i\nu}{2}}}{\pi^2(z-\bar{z})} + \nu \leftrightarrow -\nu\right). \qquad (2.5.28)
\end{aligned}$$

Using (2.5.4) and performing the radial and y integrations we find

$$\begin{aligned}
\tilde{\Omega}_{i\nu}(t_0, z_0) = &\int dy\frac{\nu z_0^{2\left(\Delta_\phi-1\right)}\left(2t_0+iy\right)^{-\Delta_\phi+\frac{i\nu}{2}+1}\left(2t_0\left(z_0^2+1\right)-iy\left(z_0^2-1\right)\right)^{-\Delta_\phi-\frac{i\nu}{2}}}{2\pi^2\left(\nu+2i\left(\Delta_\phi-1\right)\right)} \\
&\times {}_2F_1\left(-\Delta_\phi+\frac{i\nu}{2}+1, \Delta_\phi+\frac{i\nu}{2}; -\Delta_\phi+\frac{i\nu}{2}+2; \frac{2t_0+iy}{2t_0\left(z_0^2+1\right)-iy\left(z_0^2-1\right)}\right) \\
&+ (\nu \leftrightarrow -\nu) \\
= &-\frac{\pi 2^{5-4\Delta_\phi}\Gamma\left(2\Delta_\phi-2\right)t_0^{3-2\Delta_\phi}}{\Gamma\left(\Delta_\phi-\frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+\Delta_\phi\right)}\frac{\Omega_{i\nu}(\bar{z}/z)}{\sqrt{z\bar{z}}}\Bigg|_{\vec{w},y=0} + \nu \leftrightarrow -\nu\ . \qquad (2.5.29)
\end{aligned}$$

This has two salient features. First, the smearing has the effect of replacing the partial wave $\Omega_{i\nu}$ by the partial wave evaluated at the wavepacket centers. Second, the gamma functions produced by smearing cancel the gamma functions in (2.5.27). Exactly the same thing occurred for real wavepackets in [25, 26]. The Gamma-function poles correspond to double trace operators $[\phi\phi]$ in the OPE, so by smearing we have projected out these operators. Translated into a statement about the Regge OPE, this is equivalent to (2.5.16).

[11]In this expression y and $\vec{w}$ correspond to relative coordinates $y_1 - y_2$ and $\vec{w}_1 - \vec{w}_2$. Note that the integrand is independent of the absolute coordinates which leads to a volume integral divergence that is cancelled by the smeared 2-point function $\langle\overline{W}(t_0)W(t_0)\rangle$ in the denominator.

The smearing can also be performed on the Regge amplitude $G_{Regge}(z, \bar{z})$, *i.e.*, after doing the ν-integral. This illustrates more directly which regime of cross-ratios is important to project out double-traces. An example is worked out in appendix A.4.

2.6 Smearing operators with spin

Finally, we turn to the Regge OPE of spinning operators. These also obey an equation like (2.5.16), with coefficients that depend on the polarizations. However, we won't actually derive the analogue of (2.5.16). Instead, we work with the four-point function, and just show by smearing the conformal Regge amplitude that double-trace operators are projected out. In other words, the smeared correlator is equal to the smeared stress tensor block in the Regge limit.

In [9], we showed that smearing the stress tensor block reproduces from CFT the graviton causality constraints of [5], such as the suppression of the Gauss-Bonnet term in 5d gravity, or in CFT language, the suppression of $\frac{a-c}{c}$ in large-N CFTs with a large gap. In that chapter, we assumed that smearing would project out double-trace operators and gave some physical motivation for this, but did not derive it. Our goal here is to close that gap by repeating the analysis of the previous section, but now for the spinning OPEs $TT \to T$ and $JJ \to T$, where T is the stress tensor and J is a conserved spin-1 current. This is rather technical but the basic idea is identical to the scalar case — smearing puts the fields in the Witten diagram onto geodesics, and so the smeared correlator can be computed by just smearing the stress tensor conformal block. The results, and the method, are also nearly identical to [25, 26], except we use the complex rather than real wavepackets. (It is not entirely clear why this gives the same final answer, since the smearing is over a different range of cross ratios. Presumably the two different smearings can be related by a contour deformation but we will not explore this.)

2.6.1 Setup

We restrict to $D = 4$. Our conventions for the 3-point functions $\langle JJT \rangle$ and $\langle TTT \rangle$ follow the appendices of [11]. $\langle JJT \rangle$ has two structures, with constant coefficients n_s and n_f; $\langle TTT \rangle$ has three structures, with independent coefficients n_s, n_f, and n_v. The conformal collider bounds are $n_i \geq 0$.

It was shown in [60, 61] that correlation functions with spinning external operators can be expanded in terms of spinning conformal blocks which are related to derivatives of certain scalar conformal blocks

$$\begin{aligned} G^{\mu_1 \dots \mu_l, \nu_1 \dots \nu_l} \equiv& \frac{\langle \mathcal{O}^{\mu_1 \dots \mu_l}(x_1) \mathcal{O}^{\nu_1 \dots \nu_l}(x_2) \psi(x_3) \psi(x_4) \rangle}{(x_{12}^2)^{\Delta_{\mathcal{O}}+l} (x_{34}^2)^{\Delta_\psi}} \\ =& \sum_{\Delta, J} C_{\psi\psi\mathcal{O}} \sum_k C^{(k)}_{\mathcal{O}_l \mathcal{O}_l \mathcal{O}} \hat{D}^{(k), \mu_1 \dots \mu_l, \nu_1 \dots \nu_l} g^{(k)}_{\Delta, J}(z, \bar{z}), \end{aligned} \tag{2.6.1}$$

where $\hat{D}^{(k), \mu_1 \dots \mu_l, \nu_1 \dots \nu_l}$ are differential operators acting on scalar conformal blocks $g^{(k)}_{\Delta, J}(z, \bar{z})$ with shifted scaling dimensions. Applying the methods of conformal Regge theory we will obtain the following expression for the Regge limit of the spinning correlator

$$G^{\mu_1 \dots \mu_l, \nu_1 \dots \nu_l} = G_0^{\mu_1 \dots \mu_l, \nu_1 \dots \nu_l} + \int d\nu \sum_k a^{(k)}(\nu) \hat{D}^{(k), \mu_1 \dots \mu_l, \nu_1 \dots \nu_l} (z\bar{z})^{\frac{1-j(\nu)}{2}} \Omega_{i\nu}(\bar{z}/z), \tag{2.6.2}$$

where $G_0^{\mu_1\ldots\mu_l,\nu_1\ldots\nu_l}$ contains the identity contribution to the correlation function. In what follows we will use these methods to compute correlation functions involving spin 1 conserved currents, spin 2 stress tensors and scalars as external operators.

2.6.2 $JJ\psi\psi$

Denote the normalized correlator by

$$G^{\mu,\nu} \equiv \frac{\langle J^\mu(x_1)J^\nu(x_2)\psi(x_3)\psi(x_4)\rangle}{(x_{12}^2)^{\Delta_J+1}(x_{34}^2)^{\Delta_\psi}} \; . \tag{2.6.3}$$

Assuming the leading trajectory is given by (2.5.26), the Regge amplitude (subtracting the identity) takes the form

$$\epsilon_\mu\tilde{\epsilon}_\nu\delta G^{\mu,\nu} = \int d\nu \, \sum_{k=1}^{3} a^{(k)}(\nu)\epsilon_\mu\tilde{\epsilon}_\nu\hat{D}^{(k)\mu,\nu}\frac{\Omega_{i\nu}(\bar{z}/z)}{\sqrt{z\bar{z}}}, \tag{2.6.4}$$

where the differential operators $\hat{D}^{(k)\mu,\nu}$ and coefficients $a^{(k)}(\nu)$, as well as other details of this calculation, are given in appendix A.5.

At this point we will assume that Δ_ψ is large so that we may neglect contribution from the $[\psi\psi]$ double-trace operators and perform the ν integral above by summing over the residues. Finally, we perform the smearing procedure described in section (2.5.1). We choose coordinates for the operator insertions which are related to (2.5.4) by a null inversion (2.2.21),

$$\begin{aligned} x_1 =&(t_0, iy_1, i\vec{w}_1) \\ x_2 =&(-t_0, iy_2, i\vec{w}_2) \\ x_3 =&(1+z_0^2, 1-z_0^2, \vec{0}) \\ x_4 =&(-1-z_0^2, -1+z_0^2, \vec{0}). \end{aligned} \tag{2.6.5}$$

In other words, we work directly in the shifted patch, where the wavepackets are simple complex momentum insertions. We will also choose the following for the polarization vectors

$$\begin{aligned}\epsilon_\mu =& \frac{1}{2\sigma z_0^2}\left(-1,1,-\lambda,i\lambda\right)\\ \tilde{\epsilon}_\mu =& \frac{1}{2\sigma z_0^2}\left(1,-1,\lambda,i\lambda\right).\end{aligned} \tag{2.6.6}$$

We then perform the rescaling $t_0 = \sigma t_0, y_i = \sigma y_i, \vec{w}_i = \sigma \vec{w}_i$ and take $\sigma \to 0$ limit. The resulting expression will depend only on relative coordinates $\vec{w}_{12} \equiv \vec{w}$ and $y_{12} \equiv y$ (see footnote 11). We must now evaluate

$$\int dy d\vec{w} \epsilon_\mu \tilde{\epsilon}_\nu \frac{\delta G^{\mu,\nu}}{(x_{12}^2)^{\Delta_J+1}}. \tag{2.6.7}$$

Normalizing by the smeared 2-point function to regulate divergences we find

$$\begin{aligned}&-\frac{15i\pi^3\left(\lambda^2-1\right)\left(4n_f-n_s\right)}{16(z_0^2-1)^2\sigma}-\frac{15i\pi^3\left(\lambda^2-1\right)\left(4n_f-n_s\right)}{16(z_0^2-1)^3\sigma}\\ &-\frac{15i\pi^3\left(2\lambda^2+1\right)\left(8n_f+n_s\right)}{32(z_0^2-1)\sigma}-\frac{15i\pi^3\left(2\lambda^2+1\right)\left(8n_f+n_s\right)}{32\sigma}\end{aligned} \tag{2.6.8}$$

This is identical to the expression obtaining by ignoring, from the start, all of the double-trace poles in the conformal Regge amplitude. This is the main point: the smeared correlator is equal to the smeared stress tensor conformal block.

The chaos bound [41] implies that each power of λ as $z_0 \to 1$ must be $i \times$ (*positive*) [9]. Different polarizations give different signs, so this is impossible, unless

$$4n_f - n_s = 0. \tag{2.6.9}$$

In bulk language, this implies that the non-minimal coupling between the graviton and photon must be suppressed by the string scale.

In terms of the Regge OPE, this implies a smeared OPE of JJ similar to (2.5.16), but with the coefficient on the r.h.s built from polarizations, derivatives, and the n_i. It would be nice to work this out explicitly, for example by comparison to the bulk or to the 4-point Regge amplitude, but we have not done so. The combination $4n_f - n_s$ appears as the dominant coefficient in this OPE as $z_0 \to 1$.

2.6.3 $TT\psi\psi$

Now we turn to the case of external gravitons. Let

$$G^{\mu\nu,\alpha\beta} \equiv \frac{\langle T^{\mu\nu}(x_1)T^{\alpha\beta}(x_2)\psi(x_3)\psi(x_4)\rangle}{(x_{12}^2)^{\Delta_T+2}(x_{34}^2)^{\Delta_\psi}} \; . \tag{2.6.10}$$

Assuming the leading trajectory is given by (2.5.26), subtracting the identity contribution and contracting with polarization vectors in the Regge limit we have

$$\epsilon_\mu\epsilon_\nu\tilde{\epsilon}_\alpha\tilde{\epsilon}_\beta\delta G^{\mu\nu,\alpha\beta} = \int d\nu \; \sum_{k=1}^{10} a^{(k)}(\nu)\epsilon_\mu\epsilon_\nu\tilde{\epsilon}_\alpha\tilde{\epsilon}_\beta\hat{D}^{(k)\mu\nu,\alpha\beta}\frac{\Omega_{i\nu}(\bar{z}/z)}{\sqrt{z\bar{z}}}, \tag{2.6.11}$$

where the differential operators $\hat{D}^{(k)\mu\nu,\alpha\beta}$ and the coefficients $a^{(k)}(\nu)$ are too long to write down, but are available in Mathematica format from the authors. As above, we drop the heavy $[\psi\psi]$ operators, perform the ν integral by summing over residues, then do the smearing integrals. The result is

$$\begin{aligned}&\frac{240i\pi^3\left(\lambda^4-4\lambda^2+1\right)\left(4n_f-n_s-2n_v\right)}{(z_0^2-1)^4\sigma}+\frac{120i\pi^3\left(\lambda^4-4\lambda^2+1\right)\left(4n_f-n_s-2n_v\right)}{(z_0^2-1)^5\sigma}\\&+\frac{20i\pi^3\left(3\lambda^4(14n_f-n_s-22n_v)+3\lambda^2(-41n_f+9n_s+28n_v)+27n_f-8n_s-6n_v\right)}{(z_0^2-1)^3\sigma}\\&+\frac{20i\pi^3\left(3\left(\lambda^4(6n_f+n_s-18n_v)+\lambda^2(-9n_f+n_s+12n_v)+n_f\right)-2n_s+6n_v\right)}{(z_0^2-1)^2\sigma}\\&-\frac{10i\pi^3\left(6\left(\lambda^4+\lambda^2\right)+1\right)\left(6n_f+n_s+12n_v\right)}{(z_0^2-1)\sigma}-\frac{10i\pi^3\left(6\left(\lambda^4+\lambda^2\right)+1\right)\left(6n_f+n_s+12n_v\right)}{\sigma}\end{aligned} \tag{2.6.12}$$

Once again this agrees with the stress tensor term alone. In the leading term as $z_0 \to 1$ the coefficients of powers of λ come with opposite signs and hence cannot all be positive implying the following constraints on the coefficients

$$\begin{aligned} 4n_f - n_s - 2n_v =& 0 \\ n_f - 2n_v =& 0. \end{aligned} \tag{2.6.13}$$

These are precisely the combinations corresponding to higher curvature couplings, ruled out from a gravity analysis in [5]. Thus we have confirmed the argument sketched in [9] that double trace operators can be ignored in this calculation.

CHAPTER 3

A CONFORMAL COLLIDER FOR HOLOGRAPHIC CFTS

Abstract[1]

We develop a formalism to study the implications of causality on OPE coefficients in conformal field theories with large central charge and a sparse spectrum of higher spin operators. The formalism has the interpretation of a new conformal collider-type experiment for these class of CFTs and hence it has the advantage of requiring knowledge only about CFT three-point functions. This is accomplished by considering the holographic null energy operator which was introduced in [62] as a generalization of the averaged null energy operator. Analyticity properties of correlators in the Regge limit imply that the holographic null energy operator is a positive operator in a subspace of the total CFT Hilbert space. Utilizing this positivity condition, we derive bounds on three-point functions $\langle TO_1O_2\rangle$ of the stress tensor with various operators for CFTs with large central charge and a sparse spectrum. After imposing these constraints, we also find that the operator product expansions of all primary operators in the Regge limit have certain universal properties. All of these results are consistent with the expectation that CFTs in this class, irrespective of their microscopic details, admit universal gravity-like holographic dual descriptions. Furthermore, this connection enables us to constrain various inflationary observables such as the amplitude of chiral gravity waves, non-gaussanity of gravity waves and tensor-to-scalar ratio.

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3.1 Introduction

In conformal field theory (CFT), causality of four-point functions places nontrivial constraints on CFT three-point couplings. In particular, causality in the lightcone limit leads to constraints [12, 63, 64] which are identical to the bounds obtained from the conformal collider experiment [13]. Of course, this is not a coincidence. In fact, the proof of the averaged null energy condition (ANEC) $\int T_{uu} du \geq 0$ from causality [10] made it apparent that for generic CFTs, the conformal collider set-up provides an efficient tool for deriving causality constraints.

The conformal collider set-up is a simple yet powerful thought experiment that was introduced by Hofman and Maldacena [13]. In this set-up, the CFT is prepared in an excited state by creating a localized excitation which couples to some operator $\mathcal{O}$ (with or without spin) of the CFT. This excitation propagates outwards and the response of the CFT is measured by a distant calorimeter. The calorimeter effectively measures the averaged null energy flux $\langle \int T_{uu} du \rangle$ far away from the region where the excitation was created and hence the calorimeter readings should be non-negative. This gives rise to constraints on the three-point function $\langle \mathcal{O} T \mathcal{O} \rangle$, where T is the stress tensor operator. Recently, the conformal collider set-up was extended to study interference effects, leading to new bounds on OPE coefficients [65, 66].[2] All of these causality constraints are valid for every CFT in $D \geq 3$, however, additional assumptions about the CFT can lead to stronger constraints. In particular, similar logic in certain class of CFTs can shed light on how gravity emerges from CFT.

[2] Similar method was also used by [67] to constrain parity violating CFTs in $D = 3$.

Holographic CFTs

The low energy behavior of gravitons, in any sensible theory of quantum gravity, is described by the Einstein-Hilbert action plus higher derivative correction terms. However, these higher derivative terms can lead to causality-violating propagation in nontrivial backgrounds [21, 57, 68]. Requiring the theory to be causal in shockwave states, as shown by Camanho, Edelstein, Maldacena, and Zhiboedov [5] (CEMZ), does impose strong constraints on gravitational three-point interactions. For example, causality dictates that the graviton three-point coupling should be universal in quantum gravity [5] – a claim consistent with constraints obtained from unitarity and analyticity [69]. Furthermore, the AdS/CFT correspondence [2, 70, 71] immediately suggests that in any CFT with a holographic dual description, certain three-point functions (for example $\langle TTT \rangle$) must also have specific structures.

Over the past several years, it has become clear that a large class of CFTs, with or without supersymmetry, exhibits gravity-like behavior [18–20, 33–36, 42, 52, 72–88]. More recently, the CEMZ causality constraints have been derived from the CFT side for dimension $D \geq 3$ [9, 25, 26, 62, 66], under the assumptions:

- The central charge c_T is large[3]: $c_T \gg 1$
- A sparse spectrum: the lightest single trace operator with spin $\ell > 2$ has dimension

$$\Delta_{\text{gap}} \gg 1 \ .$$

[3] c_T is the coefficient of the stress tensor two-point function (see equation (A.6.7)). For gauge theories, the large c_T limit is equivalent to the large-N limit.

All of these observations indicate that CFTs in this class, irrespective of their microscopic details, admit a universal gravity-like holographic dual description at low energies. Furthermore, this connection provides us with a powerful tool to constrain gravitational interactions by studying CFTs with a large central charge and a sparse spectrum. In this chapter, we intend to adopt this point of view. First, for CFTs in this universality class (henceforth denoted *holographic CFTs*), we will derive general constraints on CFT three-point functions from causality. In light of the AdS/CFT correspondence, these CFT causality constraints translate into constraints on the low energy gravitational effective action from UV consistency.

The CEMZ causality constraints for CFTs with large central charge and a sparse spectrum were first derived in [9] from causality of the four-point function $\langle\psi\psi T_{\alpha\beta}T_{\gamma\delta}\rangle$ in the Regge limit, where ψ is a heavy scalar operator. The derivation heavily relied on the fact that the stress tensor operators in the correlator were smeared in a specific way that projected out $[TT]$ double trace contributions to the Regge correlator. The same constraints were also derived in [25, 26] by imposing unitarity on a differently smeared correlator $\langle\psi\psi T_{\alpha\beta}T_{\gamma\delta}\rangle$ in the Regge limit. Moreover, this approach was recently extended to study a mixed system of four-point functions in the Regge limit yielding new bounds on the OPE coefficients of low spin operators in holographic CFTs [66]. From the dual gravity perspective, all of these set-ups are probing local high energy scattering deep in the bulk. However, the actual CFT analysis involves computations of CFT four-point functions of spinning operators using the conformal Regge theory [32], which is technically challenging even in the large central charge limit. One might hope that in the Regge limit causality of CFT four-point functions can be translated to some holographic energy condition which is a generalization of the averaged null energy condition for holographic CFTs. Such an energy condition was recently derived in [62]. In this chapter, we will exploit this energy condition to design a new experiment, similar to the conformal collider experiment of [13], for holographic CFTs which will allow us to bypass the conformal Regge theory.

Holographic null energy condition

In the Regge limit, causality dictates that the *shockwave operator* $\int h_{uu}du$ must be non-negative for CFTs with large central charge and a sparse spectrum [62]. This immediately allows us to imagine an "AdS collider" where the boundary CFT is again prepared in the Hofman-Maldacena state $|HM\rangle$. But now the measuring device is in the bulk and measures $\langle HM|\int h_{uu}du|HM\rangle \geq 0$ (see figure 3.4). It is obvious that this set-up will reproduce all of the causality constraints, however, both technically and conceptually this is not very satisfying for several reasons. First, this correlator should be computed using Witten diagrams which is difficult when the state $|HM\rangle$ is prepared using spinning operators. Second, in the CFT language, this set-up is not illuminating because the operator $\int h_{uu}du$ has a complicated decomposition into CFT operators which consists of the stress tensor and an infinite tower of double trace operators.

In this chapter, we consider the stress tensor component of the shockwave operator [62]

$$\mathbf{E}_r(v) = \int_{-\infty}^{+\infty} du' \int_{\vec{x}^2 \leq r^2} d^{d-2}\vec{x} \left(1 - \frac{\vec{x}^2}{r^2}\right) T_{uu}\left(u', v, i\vec{x}\right) \ ,$$

which we will refer to as the *holographic null energy operator*.[4] Causality of CFT four-point function in the Regge limit [62] implies that the expectation value of the

[4] u and v are the null coordinates.

holographic null energy operator is positive in a large subspace of the total Hilbert space of holographic CFTs. Note that this operator is the averaged null energy operator smeared over a finite sphere along the imaginary transverse directions. Of course, the positivity of the holographic null energy operator is not implied by the ANEC because of the imaginary transverse directions. In fact, this operator, in general, is not positive.

A key ingredient of the positivity argument is that there exists a class of states $|\Psi\rangle$ which projects out certain double trace contributions to $\int h_{uu} du$. This is an extension of the observations made in [62].[5] These states, as we will show, are equivalent to the Hofman-Maldacena state $|HM\rangle$ which will allow us to introduce a new formalism to study causality constraints. Our formalism can be interpreted as a new collider-type experiment for holographic CFTs (see figure 3.1). Consider a CFT with large central charge and a sparse spectrum in D-dimensions. The CFT is prepared in the excited state $|HM\rangle$ by inserting a spinning operator O near the origin and an instrument measures the holographic null energy far away from the excitation:

$$\mathbf{E}(\rho) = \lim_{R\to\infty} R^2 \langle HM|\mathbf{E}_{r=\sqrt{\rho}R}(R)|HM\rangle \ .$$

The holographic null energy condition implies that $\mathbf{E}(\rho)$ is a positive function for $0 < \rho < 1$. The parameter ρ is a measure of the angular size of our measuring device at the origin and the parameter ρ can be tuned by changing the size of the device. In the gravity language, ρ plays the role of the bulk direction. In particular, $\rho \to 0$ represents the lightcone limit (AdS boundary) and hence in this limit, this set-up is equivalent to the original conformal collider experiment. On

[5]We should note that in this chapter we will not provide a general technical proof of the observation made in [62] about double trace contributions. However, we will argue that the same statement about double trace contributions is true in general.

the other hand, we are interested in probing high energy scattering deep in the bulk of the dual geometry which corresponds to the limit $\rho \to 1$.

Our conformal collider set-up has several advantages over previous methods [9, 25, 26, 62, 66]. First, we do not need to compute conformal Regge amplitudes. In our set-up, all of the constraints are directly obtained from CFT three-point functions which are fixed by conformal symmetry up to a few constant coefficients – a simplification which enables us to derive constraints in a more systematic way. Finally, our approach connects causality constraints in the Regge limit with the holographic null energy condition. This is reminiscent of the ANEC which relates causality in the lightcone limit with entanglement. So, the appearance of the holographic null energy condition perhaps is an indication of some deeper connection between boundary entanglement and bulk locality. Moreover, the recent generalization of the ANEC to continuous spin [89] suggests that there might also be a generalization of the holographic null energy condition to continuous spin.

Summary of results

The formalism that we developed in this chapter efficiently computes the expectation value of $\mathbf{E}_r$ in states $|\Psi\rangle$, constructed by inserting spinning operators.[6] This formalism involves performing certain integrals over CFT three-point functions which are fixed up to OPE coefficients by symmetries. Furthermore, we decompose the results into independent bounds corresponding to representations under spatial rotations. The inequalities following from these bounds lead to surprising equalities among the various OPE coefficients involving spinning operators and

[6]This formalism can easily be adapted to computing the contribution of any conformal multiplet to the Regge limit of four-point correlation functions.

the stress-tensor.

The first set of constraints are obtained by considering expectation values in states constructed from a single low spin operator ($\ell \leq 2$). The second set of constraints follows from the interference effects in our collider. In particular, positivity of the holographic null energy operator in states created by superposition of smeared local operators O_1 and O_2 leads to a bound on the off-diagonal matrix elements of the operator $\mathbf{E}$:

$$|\mathbf{E}_{O_1 O_2}(\rho)|^2 \leq \mathbf{E}_{O_1 O_1}(\rho)\mathbf{E}_{O_2 O_2}(\rho) \ .$$

Let us now summarize the resulting constraints for all single trace low spin ($\ell \leq 2$) operators in a holographic CFT (in $D \geq 3$).

- All three-point functions of the form $\langle TOO \rangle$ are completely fixed by the two-point function $\langle OO \rangle$.
- All three-point functions $\langle TO_1O_2 \rangle$, where O_1 and O_2 are different operators, are suppressed by Δ_{gap}.[7]

These constraints encompasses, and generalizes, all known causality constraints as obtained in [9,25,26,62,66] by studying various four-point functions in holographic CFTs. Moreover, after imposing these causality constraints, we find that the expectation value of the holographic null energy operator is universal and it is completely determined by the lightcone limit result. This observation suggests the following conclusion about the operator product expansions in holographic CFTs:

[7]There is a caveat. Our argument does not necessarily hold if scaling dimensions of O_1 and O_2 coincide with the scaling dimension of double-trace operators (at leading order in c_T). For more discussion see [65,66].

- The operator product expansion of any two smeared primary single trace operators (with or without spin) in the Regge limit is given by a *universal shockwave operator*:

$$\Psi^*[O_1]\Psi[O_2] \approx \langle \Psi^*[O_1]\Psi[O_2]\rangle - 2iE_{O_1O_2}\int_0^\infty dt\ t^2 h_{z+t\ z+t}\ ,$$

  where, $E_{O_1O_2}$ is the matrix element of the total energy operator. The operators O_1 and O_2 are smeared in such a way that they can create states which belong to the class $|\Psi\rangle$ (see section 3.3). On the right hand side, the spherical shockwave operator is written as an integral of the metric perturbation over a null geodesic: $z = t$ (where z is the bulk direction and t is the Lorentzian time) in AdS$_{d+1}$ for $D \geq 3$.

In the gravity language, the above CFT constraints translate into the statement that all higher derivative interactions in the low energy effective action must be suppressed by the new physics scale. Furthermore, in agreement with the proposal made by Meltzer and Perlmutter in [66], we find that in $D \geq 4$ CFT dual of a bulk derivative is $1/\Delta_{\text{gap}}$. However, we also notice that in $D = 3$ there is a logarithmic violation of this simple relationship between the bulk derivative and Δ_{gap}.

As a simple example of the above bounds, we derive "$a \approx c$" type relations between conformal trace anomalies in $D = 6$. In $D = 6$, there are four Weyl anomaly coefficients a_6, c_1, c_2, c_3, however, three of them (c_1, c_2, c_3) are determined by the stress tensor three-point function $\langle TTT\rangle$. Our bounds immediately imply that the anomaly coefficients must satisfy $c_1 = 4c_2 = -12c_3$. These relations between c_1, c_2, c_3 are exactly what is expected for $(2,0)$ supersymmetric theories, both holographic and non-holographic [90]. This is reminiscent of the Ooguri-Vafa conjecture [91] which states that holographic duality with low energy description in term of the Einstein gravity coupled to a finite number of matter fields exists only for supersymmetric theories.

Finally, as a new application of the holographic null energy condition, we constrain various inflationary observables such as the amplitude of chiral gravity waves, nongaussanity of gravity waves and tensor-to-scalar ratio. Our argument parallels the argument made by Cordova, Maldacena, and Turiaci in [65]. The bounds on higher curvature interactions in AdS_4 strongly suggests that these higher curvature terms should also be suppressed by the scale of new physics in the effective action in de Sitter space. Hence, any effect that arises from these higher curvature terms must be vanishingly small. For example, in $(3+1)-$dimensional gravity all parity odd interactions appear in higher derivative order. Therefore, all inflationary observables that violate parity including chiral gravity waves and parity odd graviton nongaussanity, must be suppressed by the scale of new physics. Furthermore, any detection of these effects in future experiments will imply the presence of an infinite tower of new particles with spins $\ell > 2$ and masses comparable to the Hubble scale.

Outline

The rest of the chapter is organized as follows. In section 3.2, we discuss the conformal collider set-up for holographic CFTs and review the holographic null energy condition. Then in section 3.3, we summarize our causality constraints as a statement about Regge OPE of smeared operators. In this section, we also propose a relation that connects the Regge limit with the lightcone limit for holographic CFTs. In section 3.4, we present a systematic approach of calculating

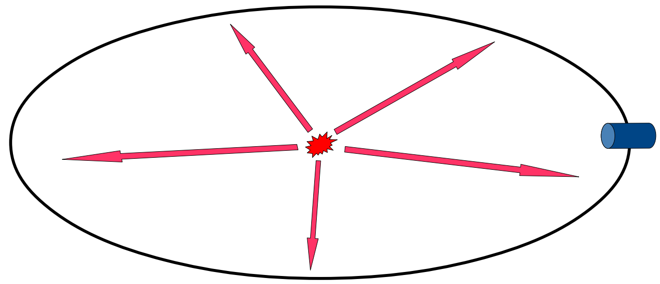


Figure 3.1: Conformal collider experiment: A localized excitation is created in a holographic CFT and an instrument which is shown in blue, measures the holographic null energy $\mathbf{E}_r$ far away from the excitation.

the expectation value of the holographic null energy operators in states created by smeared operators. This section mainly contains technical details, so it can be safely skipped by casual readers. In sections 3.5 and 3.6, we derive explicit constraints on CFT three-point functions for $D \geq 4$. The $D = 3$ case is more subtle and hence we treat it separately in section 3.7. In section 3.8, we discuss the cosmological implications of our CFT bounds. Finally, we end with concluding remarks in section 3.9.

3.2 Causality and conformal collider physics

In the lightcone limit, causality dictates that the averaged null energy operator $\int T_{uu} du$ should be non-negative [10].[8] The ANEC immediately leads to positivity of all CFT three-point functions which have the form: $\langle O| \int T_{uu} du |O\rangle \geq 0$. On the

[8]The averaged null energy condition for interacting quantum field theories in Minkowski spacetime was first derived in [46] from monotonicity of relative entropy.

other hand, for CFTs with large central charge and a sparse spectrum, causality of four-point functions in the Regge limit leads to stronger constraints. However, all of these causality conditions involve computations of CFT four-point functions of spinning operators using the conformal Regge theory [32]. The causality of CFT four-point functions even in the Regge limit can be translated to positivity of certain (holographic) energy operator [62]. In this section, with the help of that positivity condition, we develop a new conformal collider set-up enabling us to derive causality bounds directly from three-point functions.

3.2.1 A collider for holographic CFTs

We will use the following convention for points $x \in \mathbb{R}^{1,d-1}$:

$$x = (t, x^1, \vec{x}) \equiv (u, v, \vec{x}) \ , \qquad \text{where,} \qquad u = t - x^1 \ , \qquad v = t + x^1 \ . \tag{3.2.1}$$

Let us now define the holographic null energy operator:

$$\mathbf{E}_r(v) = \int_{-\infty}^{+\infty} du' \int_{\vec{x}^2 \leq r^2} d^{d-2}\vec{x} \left(1 - \frac{\vec{x}^2}{r^2}\right) T_{uu}\left(u', v, i\vec{x}\right) \ . \tag{3.2.2}$$

The holographic null energy operator is a generalization of the averaged null energy operator which was first introduced in [62].[9] In particular, in the limit $r \to 0$, this operator is equivalent to the averaged null energy operator. The kernel in (4.3.8) is positive and hence one might expect that the operator $\mathbf{E}_r(v)$ should also be positive. However, this is not true because the stress tensor is also integrated over imaginary transverse coordinates and in general $\int du' T_{uu}\left(u', v, i\vec{x}\right)$ can have either sign.

[9] Also see [39, 40] for a connection between the holographic null energy operator and the modular Hamiltonian.

Let us now carry out a collider physics thought experiment similar to [13] but with a holographic CFT in D-dimensions where $D \geq 3$ (see figure 3.1). We prepare the CFT in an excited state by inserting a spinning operator O near the origin[10]:

$$|\Psi\rangle = \int dy^1 d^{d-2}\vec{y}\, \epsilon.O(-i\delta, y^1, \vec{y})|0\rangle \ , \tag{3.2.3}$$

where, ϵ is the polarization of the operator O and $\delta > 0$. Similarly,

$$\langle\Psi| = \int dy^1 d^{d-2}\vec{y}\, \langle 0|\epsilon^*.O(i\delta, y^1, \vec{y}) \ . \tag{3.2.4}$$

The state $|\Psi\rangle$ is equivalent to the Hofman-Maldacena state of the original conformal collider experiment [13]. Now we imagine an instrument that measures the holographic null energy $\mathbf{E}_r(v)$ far away from the excitation:

$$\mathbf{E}(\rho) = \lim_{B\to\infty} \langle\Psi|\mathbf{E}_{\sqrt{\rho}B}(B)|\Psi\rangle \ , \tag{3.2.5}$$

where, $0 < \rho < 1$. The parameter ρ is a measure of the size of the measuring device which we can tune. The measuring device is placed at a distance B away from the excitation and the angular size of the device is roughly $\rho^{\frac{d-2}{2}}$. A priori it is not obvious that the measured value $\mathbf{E}(\rho)$ has to be positive. However, later in this section, by using the positivity conditions of [62], we will show that for CFTs with large central charge and a sparse spectrum in $D \geq 3$:

$$\mathbf{E}(\rho) \geq 0 \ , \qquad 0 < \rho < 1 \tag{3.2.6}$$

for a class of states that has the form (4.3.11). This inequality will play an important role in this chapter and we will refer to this as holographic null energy condition. In the limit $\rho \to 0$, the holographic null energy operator becomes $\int du' T_{uu}(u')$ and $\mathbf{E}(\rho) \geq 0$ is true for any CFT. In this limit, the positivity of

[10] O is not necessarily a primary operator. In fact O can be a linear combination of various operators with different spins. Also note that in equation (4.3.11), $\epsilon.O \equiv \epsilon_{\mu\nu\dots}O^{\mu\nu\dots}$.

$\mathbf{E}(\rho)$ reproduces the conformal collider bounds of [10, 13, 65–67]. Note that the wavepacket of [13] is implemented here by the order of limits. We first perform the u'-integral in (3.2.5) and then take the limit $B \to \infty$. The same trick was used in [10] to derive conformal collider bounds directly from a Rindler reflection symmetric set-up.

This conformal collider set-up is equivalent to the set-up used in [9,62], however, now we do not need to compute a four-point function. For example, in $D = 4$, if we take O to be the stress tensor and choose the polarization $\epsilon^\mu = (-i, -i, i\lambda, \lambda)$, as we demonstrate in appendix A.9, each power of λ should individually satisfy (4.3.10). In particular, in the limit $\rho \to 1$, we recover $a = c$ from (4.3.10).

Before we proceed, let us rewrite (3.2.5) in a more familiar form. The Hofman-Maldacena state of the original conformal collider experiment [13] is given by

$$|HM\rangle = \int dtdy^1 d^{d-2}\vec{y} e^{-(t^2+(y^1)^2+\vec{y}^2)/D} e^{-i\omega t} \epsilon^{\mu\nu\cdots} O_{\mu\nu\ldots}(t, y^1, \vec{y}) \ , \qquad \omega D \gg 1 \ . \tag{3.2.7}$$

Then (4.3.10) immediately implies that

$$\lim_{R\to\infty} R^2 \langle HM|\mathbf{E}_{r=\sqrt{\rho}R}(R)|HM\rangle \geq 0 \ . \tag{3.2.8}$$

3.2.2 Holographic null energy condition

It was shown in our previous paper [62] that causality of CFT four-point functions in the Regge limit implies positivity of certain smeared CFT three-point functions. First, we review and further explore that positivity condition. Then, we derive (4.3.10) as a simple consequence.

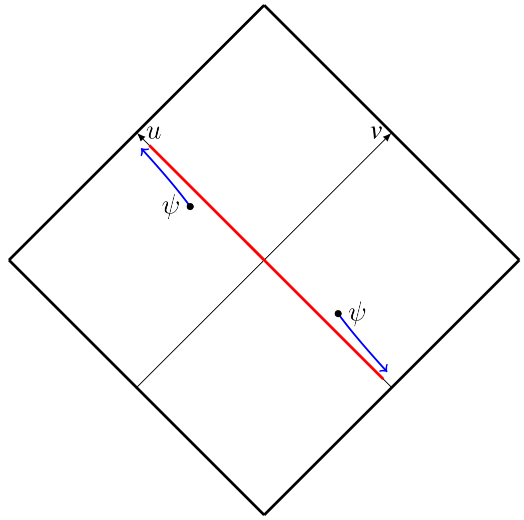


Figure 3.2: In the Regge limit the leading correction to the $\psi\psi$ OPE is the graviton h_{uu} integrated over the red line.

Regge limit and OPE of heavy scalars

We start with a discussion on the Regge OPE of heavy operators in the holographic limit. Let us consider a real scalar primary ψ in a $D-$dimensional CFT with $\Delta_\psi \gg 1$. In general, one can replace any two nearby operators by their OPE. For example, $\psi(u,v)\psi(-u,-v)$ can be written as[11]

$$\psi(u,v)\psi(-u,-v) = \sum_p C_p(u,v;\partial_u,\partial_v)\mathcal{O}_p(0,0) \ , \tag{3.2.9}$$

where, the sum is over all primaries. In a generic CFT, the lightcone and the Regge limits of a correlator are controlled by different sets of operators. In the standard lightcone limit $v \to 0$ (with u fixed), the above OPE can be organized as an expansion in twist $\tau_p = \Delta_p - \ell_p$ (Δ is scaling dimension and ℓ is spin) which leads to a simple lightcone OPE [10]. On the other hand, the Regge limit

[11] Whenever we drop some spacetime coordinates, those coordinates are set to zero.

is obtained by taking (see figure 3.2)

$$v \to 0\ , \qquad u \to \infty\ , \qquad uv = \text{fixed}\ . \tag{3.2.10}$$

Unlike the lightcone limit, the Regge limit gets significant contributions from high spin exchanges. Even when the central charge c_T (defined in (A.6.7)) is large, complication arises because an infinite tower of double trace operators become relevant in the Regge limit. However, under the additional assumption that the spectrum of single trace operators with $\ell > 2$ is sparse, simplification emerges and the Regge OPE can be written as [62]

$$\frac{\psi(u,v)\psi(-u,-v)}{\langle\psi(u,v)\psi(-u,-v)\rangle} = 1 - \frac{\Delta_\psi u}{2}\int_{-\infty}^{\infty} du' h_{uu}(u', v' = 0, \vec{x}' = 0, z' = \sqrt{-uv}) + \cdots\ , \tag{3.2.11}$$

where, $c_T \gg \Delta_\psi \gg 1$ and dots are $\mathcal{O}(u^0, \Delta_\psi^0, 1/c_T^2)$ terms. h_{uu} in the above equation is the bulk metric perturbation in AdS$_{d+1}$ (where z is the bulk coordinate) which is integrated over a null geodesic. In the gravity language, contributions of an infinite tower of primary operators translate into a single term because the dominant contribution to the four-point function comes from the Witten diagram with a single graviton exchange. Hence, the right hand side of (3.2.11) should be thought of as a CFT operator written in terms of the bulk metric. In particular, $\int du h_{uu}$ contains the stress tensor $T_{\mu\nu}$ as well as all double trace operators $[O_1 O_2]$ built from the light operators in theory, e.g., $[TT]$ double trace operators. The stress tensor contribution of $\int du h_{uu}$ can be computed using the HKLL prescription for h_{uu} [38].

Causality of the Regge correlator dictates that the operator $\int du h_{uu}$ has to be positive [62] and hence any three-point function which has the form $\langle O| \int h_{uu} du |O\rangle$ must be positive as well. From the CFT perspective, this positivity condition both

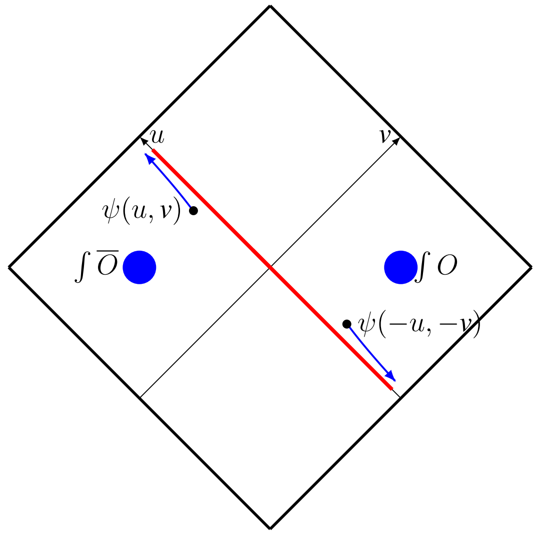


Figure 3.3: Kinematics for the derivation of the holographic null energy condition. Operators Os are smeared over some regions in a Rindler reflection symmetric way.

technically and conceptually is not very useful. However, we will show that for specific states, only the stress tensor contribution of $\int du h_{uu}$ is important which will lead us to the holographic null energy condition. Before we proceed, let us note that the contribution of the single trace stress tensor and its derivatives to the Regge OPE (3.2.11) can be written in terms of the holographic null energy operator [62]

$$\frac{\psi(u,v)\psi(-u,-v)|_T}{\langle\psi(u,v)\psi(-u,-v)\rangle} = -\frac{\Delta_\psi 2^d \pi^{\frac{1}{2}-d}\Gamma(\frac{d+2}{2})\Gamma(\frac{d+3}{2})u}{c_T(d-1)}\mathbf{E}_{r=\sqrt{-uv}}(0) \ , \tag{3.2.12}$$

where, $\mathbf{E}_r(v)$ is defined in (4.3.8).

Positivity

Consider a Rindler reflection symmetric four-point function

$$G = \frac{\langle \overline{\varepsilon.O(B)}\psi(u,v)\psi(-u,-v)\varepsilon.O(B)\rangle}{\langle \overline{\varepsilon.O(B)}\varepsilon.O(B)\rangle\langle\psi(u,v)\psi(-u,-v)\rangle} \ , \tag{3.2.13}$$

in the regime (3.2.10), as shown in figure 3.3. $\varepsilon.O(B)$ is an arbitrary operator with or without spin (not necessarily a primary operator) smeared over some region:

$$\varepsilon.O(B) = \int d\tau d^{d-2}\vec{y}\, \varepsilon.O(t = i(B+\tau), y^1 = \delta, \vec{y}) \ , \tag{3.2.14}$$

where, $\delta > 0$ and ε is the polarization (when O is a spinning operator). Operator $\overline{\varepsilon.O}$ is the Rindler reflection of the operator O (see [10] for a detailed discussion):

$$\overline{\varepsilon.O(B)} = \int d\tau d^{d-2}\vec{y}\, \overline{\varepsilon}.O^{\dagger}(t = i(B+\tau), y^1 = -\delta, \vec{y}) \ , \tag{3.2.15}$$

where, the Hermitian conjugate on the right-hand side does not act on the coordinates. $\overline{\varepsilon}$ is the Rindler reflection of the polarization ε:

$$\overline{\varepsilon^{\mu\nu\cdots}} \equiv (-1)^P (\varepsilon^{\mu\nu\cdots})^* \tag{3.2.16}$$

where P is the number of t-indices plus y^1-indices.

Following [9], let us define

$$u = \frac{1}{\sigma} \ , \qquad v = -\sigma B^2 \rho \tag{3.2.17}$$

with $B > 0, \sigma > 0$ and $0 < \rho < 1$. The Regge limit is obtained by taking $\sigma \to 0$ with ρ, B fixed. Now using the OPE (3.2.11), we obatin

$$G \equiv 1 + \delta G = 1 - \frac{\Delta_\psi}{2\sigma\mathcal{N}} \langle \overline{\varepsilon.O(B)} \int du' h_{uu}(u', z = B\sqrt{\rho})\, \varepsilon.O(B)\rangle \tag{3.2.18}$$

with $\mathcal{N} = \langle\overline{\varepsilon.O(B)}\varepsilon.O(B)\rangle > 0$. The null line integral in the above expression is computed by choosing appropriate contour. We can now repeat the arguments of [9, 10] which tells us that the boundary CFT will be causal if and only if

$$\text{Im}(\delta G) \leq 0 \ , \tag{3.2.19}$$

which is precisely the chaos bound of [41]. Since, δG as obtained from (3.2.18) is purely imaginary, therefore the last inequality is equivalent to

$$i\langle\overline{\varepsilon.O(B)} \int du' h_{uu}(u', z = B\sqrt{\rho})\, \varepsilon.O(B)\rangle \geq 0 \tag{3.2.20}$$

for any operator O. After we perform a rotation by $\pi/2$ in the Euclidean $\tau - x^1$ plane, this is precisely the statement that the shockwave operator $\int du h_{uu}$ is positive [62]. This is a CFT version of the a bulk causality condition proposed by Engelhardt and Fischetti in [43]. They showed that asymptotically AdS spacetimes satisfy boundary causality if and only if metric perturbations satisfy $\int du h_{uu} \geq 0$. This requirement is weaker than the bulk null energy condition which was the starting point of the Gao-Wald theorem [44].

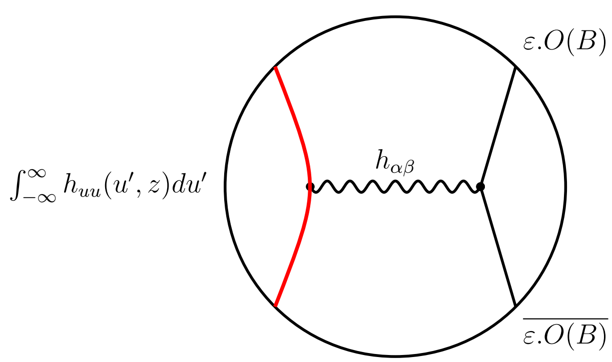


Figure 3.4: The Witten diagram for the correlator $\langle\overline{\varepsilon.O(B)} \int du' h_{uu}(u', z = B\sqrt{\rho})\, \varepsilon.O(B)\rangle$.

The three-point function (3.2.20) can be computed using the Witten diagram 3.4, however, we want a three-point function that can be computed directly in CFT. It was also shown in [62] that in the limit $B \to \infty$, the smearing projects out $[OO]$ double-trace contributions in the correlator (3.2.13) when O is a scalar operator or a spin-1 conserved current or the stress tensor T. There are plenty of evidences which suggest that the same statement is true for any operator O. We will not attempt to prove this statement, instead we conjecture that the smearing projects out $[OO]$ double-trace contributions for any operator O. The intuition comes from gravity. In the Witten diagram 3.4, the smearing puts the field dual to the operator O onto a geodesic, converting the Witten diagram 3.4 into a geodesic Witten diagram which receives contribution only from the stress tensor exchange [52]. As an immediate consequence, we can replace $\int du' h_{uu}$ in (3.2.20) by the single trace stress tensor contribution (3.2.12), yielding

$$-i \lim_{B\to\infty} \langle \overline{\varepsilon.O(B)}\, \mathbf{E}_{r=\sqrt{\rho}B}(0)\, \varepsilon.O(B)\rangle \geq 0 \ . \tag{3.2.21}$$

This is a statement about CFT three-point function which allows us to circumvent the computation of four-point functions.[12] Let us note that our conjecture about the double trace operator is simply a technical fact about the smearing that we performed. Later in the chapter, we will justify our conjecture about double trace operators by demonstrating that the inequality (3.2.21) reproduces all known causality constraints for holographic CFTs. This is a non-trivial check of the conjecture, however, one can perform a more direct check by utilizing the conformal Regge theory. It is not difficult to show case by case that δG as obtained from

[12]Let us note that if there are non-conserved spin-2 single trace primaries in the theory, they can also contribute to the four-point function in the Regge limit and hence equation (3.2.21) will not be true. For now, we assume that if they are present, they do not contribute as an exchange operator. However, as will discuss in section 3.6, this assumption is not entirely necessary.

(3.2.13) receives contribution only from the stress tensor exchange. But we admit that it will be nice to have a more general technical proof.

Let us make few comments regarding the regime of validity for the inequality (3.2.21).

- The inequality is true for any $0 < \rho < 1$ for CFTs in $D \geq 3$ with large central charge and a sparse spectrum. In particular, in the limit $\rho \to 1$, (3.2.21) probes scattering at a point deep in the interior of AdS, similar to [5,62].
- The limit $\rho \to 0$ corresponds to the lightcone limit and in this limit, the inequality is true for any interacting CFT in $D \geq 3$. Furthermore, in this limit, the inequality (3.2.21) is equivalent to the conformal collider set-up of [13] and hence it yields optimal bounds.

We will use (3.2.21) to derive constraints for holographic CFTs. So, let us rewrite (3.2.21) in a more explicit form that we will use in later sections:

$$-i\int d\tau d^{d-2}\vec{y} \lim_{B\to\infty} \langle \bar{\varepsilon}.O^\dagger(iB,-\delta,\vec{0})\,\mathbf{E}_{r=\sqrt{\rho}B}(0)\,\varepsilon.O(i(B+\tau),\delta,\vec{y})\rangle \geq 0\ . \quad (3.2.22)$$

We want to stress that in the above expression, order of limits is important. We perform the u' integral first and then take the large B limit. Also note that we are only smearing one of the operators because the other smearing integral will only give an overall volume factor. This is a consequence of the large B limit and this volume factor is the same factor that appears in the smeared two-point function.

The inequality (3.2.22) is not yet an expectation value of the holographic null energy operator in a state which has the form (4.3.11). However, we can rewrite the inequality (3.2.22) as an expectation value. First, we perform a rotation R in (3.2.22) that rotates by $\pi/2$ in the Euclidean $\tau - x^1$ plane where $\tau = it$ (see appendix A of [10]). Then we perform a translation along x^1-direction by B. This procedure converts (3.2.22) into an expectation value[13] :

$$\lim_{B\to\infty} \langle \Psi | \mathbf{E}_{\sqrt{\rho}B}(B) | \Psi \rangle \geq 0 \ , \tag{3.2.24}$$

where, $|\Psi\rangle$ is a class of states which has the form (4.3.11). This concludes the proof of the holographic null energy condition.

3.2.3 Corrections from higher spin operators

The holographic null energy condition is exact strictly in the $\Delta_{\text{gap}} \to \infty$ limit. Therefore, all of the constraints obtained from the holographic null energy condition in the limit $\rho \to 1$ will receive corrections from higher spin operators above the gap. A finite number of such operators will violate causality/chaos bound and hence this scenario is ruled out. However, it is expected that an infinite tower of new higher spin operators with $\Delta > \Delta_{\text{gap}}$ starts contributing as we approach the limit $\rho \to 1$. Let us now estimate the correction to the causality constraints if we include these higher spin operators with $\Delta > \Delta_{\text{gap}}$, where,

$$\sqrt{c_T} \gg \Delta_{\text{gap}} \gg 1 \ . \tag{3.2.25}$$

[13] We should also transform the polarization tensor accordingly (see [10]). In particular, polarizations $\epsilon^{\mu\nu\cdots}$ (as used in equation (4.3.11)) and $\varepsilon^{\mu\nu\cdots}$ (which has been used throughout the chapter whenever we have a Rindler reflection symmetric set-up) are related in the following way:

$$\epsilon^{\mu\nu\cdots} = \left(\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\cdots\right)\varepsilon^{\alpha\beta\cdots} \ , \qquad \Lambda^{\mu}_{\alpha} = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \ . \tag{3.2.23}$$

Note that if $\bar{\varepsilon}_1 = \varepsilon_2$, then $\epsilon_1^{\dagger} = \epsilon_2$.

We consider a single higher spin operator with spin ℓ and dimension $\Delta = \Delta_{\rm gap}$ and generalize the argument of our previous paper [9]. Contribution of this operator to (3.2.13) in the limit $\rho \to 1$ is given by [9]

$$\delta G \sim \frac{i}{\sigma^{\ell-1}} \frac{e^{-s\Delta_{\rm gap}/2}}{s^a} \ , \qquad s = 1-\rho \ , \tag{3.2.26}$$

where, a is a positive number and we have assumed that $\Delta_{\rm gap} \gg \ell$. Therefore, these higher spin operators becomes relevant in the strict limit of $s \to 0$. On the other hand, we can safely ignore these operators when $s \gtrsim 1/\Delta_{\rm gap}$.[14] So, we can trust the causality condition (3.2.21) as well as the collider bound (4.3.10) only in the regime $1/\Delta_{\rm gap} \lesssim s < 1$ and the strongest constraints can be obtained by setting $s \sim 1/\Delta_{\rm gap}$.

Let us now schematically write

$$\text{Im} \lim_{B\to\infty} \langle \overline{\varepsilon.O(B)}\, \mathbf{E}_{r=\sqrt{\rho}B}(0)\, \varepsilon.O(B)\rangle \sim \sum_n (\pm) \frac{t_n}{(1-\rho)^n} + \frac{c_0}{(1-\rho)^{d-3}} + \cdots \ , \tag{3.2.27}$$

where, the sum is over terms which change sign for different polarizations and hence in the absence of the higher spin operators causality condition leads to $t_n = 0$. On the other hand, we will show in the rest of the chapter that after imposing the causality constraints the leading non-vanishing term in the limit $\rho \to 1$ goes as $\frac{c_0}{(1-\rho)^{d-3}}$, where c_0 is positive.[15] Now, setting $\rho \sim 1 - 1/\Delta_{\rm gap}$, from

[14]We should note that δG has large numerical factors. Here, similar to [9], we are making an additional assumption that OPE coefficients which appear in δG are small enough to cancel these large numerical factors.

[15]In $D = 3$, the leading nonzero term goes as $-c_0 \ln(1-\rho)$ and hence the $\Delta_{\rm gap}$-correction is given by

$$\left|\frac{t_n}{c_0}\right| \lesssim \frac{\ln \Delta_{\rm gap}}{\Delta_{\rm gap}^n} \ . \tag{3.2.28}$$

the causality/chaos bound (3.2.21), we obtain

$$\left|\frac{t_n}{c_0}\right| \lesssim \frac{1}{\Delta_{\rm gap}^{n-d+3}} \ . \tag{3.2.29}$$

3.3 Universality of the smeared Regge OPE

In the rest of the chapter, we will derive constraints using the conformal collider for the holographic null energy operator. In this section, we summarize the results as a statement about the Regge OPE of smeared single trace operators with low spin. Causality of the Regge correlators suggests that the operator product expansion of any two smeared primary operators (with or without spin) of CFTs with large central charge and a sparse spectrum should approach a universal form in the Regge limit.

Let us consider two arbitrary primary single trace low spin operators O_1 and O_2 ($\ell \leq 2$). We now smear the operators following (4.3.11):

$$\Psi^*[O_1] = \int dy^1 d^{d-2}\vec{y}\, \epsilon_1^*.O_1(i\delta, y^1, \vec{y}) \ , \tag{3.3.1}$$

$$\Psi[O_2] = \int dy^1 d^{d-2}\vec{y}\, \epsilon_2.O_2(-i\delta, y^1, \vec{y}) \ , \tag{3.3.2}$$

where, ϵ_1 and ϵ_2 are polarizations of operators O_1 and O_2, respectively (when they have spins). We then perform the rescaling $\delta = \sigma\delta$, $y^1 = \sigma y^1$, and $\vec{y} = \sigma\vec{y}$ and take the limit $\sigma \to 0$. In this limit, we claim that chaos/causality bounds guarantee that the OPE of $\Psi^*[O_1]$ and $\Psi[O_2]$ (up to order $1/c_T$) is given by a universal operator $\mathbf{H}$:

$$\Psi^*[O_1]\Psi[O_2] = \langle\Psi^*[O_1]\Psi[O_2]\rangle + \langle\Psi^*[O_1]\mathbf{E}_{lc}\,\Psi[O_2]\rangle\mathbf{H} + \cdots \ , \tag{3.3.3}$$

where, dots represent terms which are suppressed by either the large gap limit or the large c_T limit or the Regge limit. And $\mathbf{E}_{lc}$ is the lightcone limit of the operator (4.3.8):

$$\mathbf{E}_{lc} \equiv \int du' T_{uu}(u', v = 1) \sim \lim_{r \to 0} \frac{\mathbf{E}_r(v = 1)}{r^{d-2}} \ . \tag{3.3.4}$$

This OPE holds if all other operator insertions are finite distance away. In general, $\mathbf{H}$ is a complicated operator which contains the stress tensor and an infinite set of double trace operators. However, the important point is that the same operator $\mathbf{H}$ appears in the OPE of all operators and does not depend on the polarizations. Only the coefficient of $\mathbf{H}$ depends on O_1 and O_2. This coefficient can be chosen to be the contribution in a regular conformal collider experiment which is determined by the lightcone limit. Also note that when O_1 and O_2 are different operators the first term in (3.3.3) vanishes, however, the second term can still be nonzero.

When O_1 and O_2 are scalar operators, (3.3.3) is a simple consequence of the smeared Regge OPE of [62]. Moreover, we are also claiming that the OPE (3.3.3) holds in the Regge limit even when O_1 and O_2 are spinning operators. However, for spinning operators, the OPE (3.3.3) is true only after we first impose chaos/causality constraints that we obtained from the holographic null energy condition

$$\lim_{B \to \infty} \langle (c_1^* \Psi^*[O_1] + c_2^* \Psi^*[O_2]) \mathbf{E}_{\sqrt{\rho} B}(B) (c_1 \Psi[O_1] + c_2 \Psi[O_2]) \rangle \geq 0 \tag{3.3.5}$$

for arbitrary c_1 and c_2. For scalar operators, the Regge correlator is trivially causal. Since the same operator $\mathbf{H}$ appears in the OPE (3.3.3) of all operators, it is obvious that the equation (3.3.3) is a sufficient condition that makes all of the Regge correlators causal. In this chapter, we will not explicitly prove that (3.3.3) is a necessary condition. Rather, in the rest of the chapter, we will show that

(3.3.3) is true for various spinning operators. Note that a hint of this property of the Regge OPE was present even in our previous paper [62].

For heavy scalar operators, the smearing integrals in (3.3.3) can be ignored because they only produce overall volume factors. Hence, for a heavy scalar $\mathcal{O}_H$, with $1 \ll \Delta_H \ll \sqrt{c_T}$ the OPE (3.3.3) is very simple. Therefore, the Regge OPE of any two smeared primary operators is determined by the OPE of two heavy scalar operators, in particular

$$\mathbf{H} = \frac{\mathcal{O}_H(i\delta)\mathcal{O}_H(-i\delta) - \langle \mathcal{O}_H(i\delta)\mathcal{O}_H(-i\delta)\rangle}{\langle \mathcal{O}_H(i\delta)\mathbf{E}_{lc}\,\mathcal{O}_H(-i\delta)\rangle}\ . \tag{3.3.6}$$

Let us now consider correlator of the holographic null energy operator with two arbitrary smeared operators $\Psi^*[O_1]$ and $\Psi[O_2]$. The equation (3.3.3) predicts that after imposing all of the causality conditions the correlator $\langle \Psi^*[O_1]\mathbf{E}_r(v)\Psi[O_2]\rangle$ can be written as a product of the lightcone answer and a correlator of the holographic null energy operator with heavy scalars. In particular, if we define

$$f_{O_1O_2}(\rho) \equiv \lim_{B\to\infty} \langle \Psi^*[O_1]\mathbf{E}_{\sqrt{\rho}B}(B)\Psi[O_2]\rangle \tag{3.3.7}$$

then it can be easily shown that equations (3.3.3) and (3.3.6) imply

$$f_{O_1O_2}(\rho) = \frac{f_{O_1O_2}(\rho\to 0) f_{\mathcal{O}_H\mathcal{O}_H}(\rho)}{f_{\mathcal{O}_H\mathcal{O}_H}(\rho\to 0)} + \cdots\ , \tag{3.3.8}$$

where, dots represent terms suppressed by Δ_{gap}. We can further simplify by computing the scalar part of the above equation, yielding

$$f_{O_1O_2}(\rho) = \lim_{\rho_0\to 0} f_{O_1O_2}(\rho_0)\left(\frac{\rho}{\rho_0}\right)^{\frac{d-2}{2}} {}_2F_1\left(\frac{d-2}{2}, d-1; \frac{d+2}{2}; \rho\right) + \cdots\ . \tag{3.3.9}$$

Broadly speaking, this equation relates UV (Regge limit) with IR (lightcone limit). It is rather remarkable that for holographic CFTs the Regge limit is completely determined by the lightcone limit. In the following sections, we will check the OPE (3.3.3) by demonstrating that the above relation holds for various operators with or without spin.

3.3.1 Gravity interpretation

The Regge OPE (3.3.3) has a nice gravity interpretation. The operator $\mathbf{H}$ is a complicated CFT operator, however, when written in terms of the bulk metric it has a simple expression. In particular, in the gravity language the Regge OPE (3.3.3) can be rewritten as[16]

$$\Psi^*[O_1]\Psi[O_2] = \langle \Psi^*[O_1]\Psi[O_2]\rangle - 2iE_{O_1O_2}\int_0^\infty t^2 h_{z+t\ z+t}(z=t,t)dt\ , \qquad (3.3.10)$$

where, $E_{O_1O_2}$ is the matrix element of the total energy operator $\langle \Psi^*[O_1]\mathbf{E}\Psi[O_2]\rangle$. On the right hand side the operator $\mathbf{H}$ is now written as the bulk metric perturbation integrated over a null geodesic $z=t, y^1=0, \vec{y}=\vec{0}$ in AdS_{d+1}. Therefore, $\mathbf{H}$ is a *shockwave operator* that creates a spherical shockwave in AdS.

The OPE (3.3.10) has been derived by starting from the planar shockwave operator of [62]. In the gravity language, the OPE of heavy scalars $\mathcal{O}_H(i\delta)\mathcal{O}_H(-i\delta)$ can be obtained from the Regge OPE of [62] by performing the following change of coordinates:

$$u \to \frac{z_0^2}{u}\ , \qquad v \to -v + \frac{\vec{y}^2}{u} + \frac{z^2}{u}\ , \qquad \vec{y} \to \frac{z_0\vec{y}}{u}\ , \qquad z \to \frac{zz_0}{u}\ , \qquad (3.3.11)$$

where, z_0 is the position of the planar shockwave operator in [62]. On the boundary this change of coordinates acts as a conformal transformation. On the other hand,

[16] $h_{z+t\ z+t}$ is defined in the usual way: $h_{z+t\ z+t} = \frac{1}{4}(h_{tt} + 2h_{tz} + h_{zz})$.

in the bulk this change of coordinate converts the planar shockwave operator into the spherical shockwave operator. Now the universality of the Regge OPE immediately implies that the same spherical shockwave operator will also appear in (3.3.10).

It is important to note that it is not surprising that the smeared Regge OPE can be expressed as an integral over a geodesic. After all, this has already been shown in [62] for light scalar operators. Moreover, our conjecture about double trace contributions implies the same for any primary single trace operator. However, the non-trivial consequence of the HNEC is the appearance of the same spherical shockwave operator in the OPE (3.3.10) for all single trace operators. This universality of the Regge OPE can be interpreted as the CFT version of the equivalence principle in the bulk.

The form of the OPE (3.3.10) is fixed by the conformal symmetry and causality of the boundary CFT and in the dual gravity language, it has an interesting consequence. First, consider a single light operator O_1 with spin $\ell \leq 2$. The OPE (3.3.10) implies that one can create a spherical shockwave in the bulk by inserting the smeared operator $\Psi[O_1]$. The resulting shockwave has an energy $\sim E_{O_1 O_1}$ and the bulk metric is identical to the shockwave geometry obtained from an infinitely boosted AdS-Schwarzschild black hole [16]. Furthermore, the form of the OPE (3.3.10) also dictates that this process of creating bulk shockwaves obeys a simple *superposition principle*. Consider an operator $\mathcal{O}$ which is a linear combination of several low spin operators

$$\mathcal{O} = c_1 O_1 + c_2 O_2 + c_3 O_3 + \cdots \ . \tag{3.3.12}$$

The smeared operator $\Psi[\mathcal{O}]$ again creates a spherical shockwave in the bulk but now with an energy $\sim E_{\mathcal{O}\mathcal{O}}$. Therefore, causality of four-point functions of the boundary CFT translates into a shockwave superposition principle in the bulk.

3.4 Nitty-gritty of doing the integrals

The aim of the rest of the chapter is to derive constraints by evaluating (3.2.21) for various spinning operators. So, in this section we present a systematic approach of calculating (3.2.21). As an example, we will explicitly show the computation of (3.2.21) for scalars which can be easily generalized for spinning operators. Then, we briefly sketch the calculation for the spinning case. This section consists of technical details, so casual readers can skip this section.

Let us now introduce the notation:

$$\mathbf{E}_{O_1O_2}(\rho) \equiv -i \lim_{B\to\infty} \langle \overline{\varepsilon_1.O_1(B)}\, \mathbf{E}_{r=\sqrt{\rho}B}(0)\, \varepsilon_2.O_2(B)\rangle \ , \tag{3.4.1}$$

where, $\varepsilon.O(B)$ and $\overline{\varepsilon.O(B)}$ are defined in (3.2.14) and (3.2.15) respectively. $\mathbf{E}_{O_1O_2}(\rho)$ is a positive function when O_1 and O_2 are the same operators and $\varepsilon_1 = \varepsilon_2$. This positivity is equivalent to the holographic null energy condition (4.3.10):

$$\mathbf{E}_{OO}(\rho) = \mathbf{E}(\rho) \geq 0 \ . \tag{3.4.2}$$

The function $\mathbf{E}_{O_1O_2}(\rho)$ is also closely related to $f_{O_1O_2}(\rho)$ as defined in (3.3.7). However, there is a key difference: $\mathbf{E}_{O_1O_2}(\rho) = f_{O_1O_2}(\rho)$ only after we impose causality constraints on $\mathbf{E}_{O_1O_2}(\rho)$.

Let us now evaluate $\mathbf{E}_{O_1O_2}(\rho)$:

$$\mathbf{E}_{O_1O_2}(\rho) = -\frac{iB^{d-2}}{\rho}\lim_{B\to\infty}\int_{-\infty}^{\infty}d\tilde{u}\int_{\vec{x}_3^2\leq\rho}d^{d-2}\vec{x}_3\int d\tau d^{d-2}\vec{y}\,(\rho-\vec{x}_3^2)$$
$$\times\langle\overline{\varepsilon_1}.O_1(iB,-\delta,0)T_{uu}(\tilde{u},0,iB\vec{x}_3)\varepsilon_2.O_2(i(B+\tau),\delta,\vec{y})\rangle\ , \quad (3.4.3)$$

where, we have rescaled $\vec{x}_3$ to $B\vec{x}_3$ so that the bounds of integration becomes $\vec{x}_3^2\leq\rho$.[17] Note that we are only smearing one of the operators because the other smearing integral will only give an overall volume factor. So, the computation of $\mathbf{E}_{O_1O_2}(\rho)$ is reduced to performing certain integrals over a CFT 3-point function whose form is fixed by conformal invariance up to constant OPE coefficients.

Order of limits:

The expression (3.4.3) is evaluated by first performing the $\tilde{u}$-integral using an appropriate contour. Then we take the $B\to\infty$ limit, yielding a relatively simple expression. To perform the smearing integrals, it is convenient to package τ and $\vec{y}$ together in a $(d-1)$-dimensional vector $\vec{\mathbf{k}}$. The resulting expression can be written covariantly by decomposing the D-dimensional vectors x_i and polarization vectors ε_i into scalars and $(d-1)$-vectors under rotations in $(\tau$-$\vec{y})$-space that is $\mathbb{R}^{d-1}$. The smearing integrals can then be performed in a covariant way using familiar techniques used in Feynman diagram computations. Finally we perform the $(d-2)$-dimensional integral over $\vec{x}_3$. Note that we have exchanged the order in which we perform integrations.

The advantage of this method is that the spin and scaling dimension of the

[17]For the sake of clarity let us again note that positions of operators O_1 and O_2 in (3.4.3) are labelled by $(t,x^1,\vec{x})$. Whereas, position of the stress tensor operator in (3.4.3) is labelled by $(u,v,\vec{x})$.

external and exchanged operators as well as the spacetime dimensions are simply constant parameters in the integrand and the integrals can in principle be performed for arbitrary values resulting in general expressions as functions of these parameters.

3.4.1 Scalar operators

As a demonstration of the formalism in action we will now compute (3.4.3) for scalar operators. The three point function of interest in this case is entirely fixed by conformal invariance [6]

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)T_{\mu\nu}(x_3)\rangle = \frac{C_{\mathcal{O}\mathcal{O}T} I_{\mu\nu}}{x_{23}^{d-2} x_{12}^{2\Delta_{\mathcal{O}}+2-d} x_{13}^{d-2}} \ , \tag{3.4.4}$$

where,

$$x_{IJ} = |x_I - x_J| \ , \qquad I^{\mu\nu} = \left(\frac{x_{13}^{\mu}}{x_{13}^2} - \frac{x_{23}^{\mu}}{x_{23}^2}\right)\left(\frac{x_{13}^{\nu}}{x_{13}^2} - \frac{x_{23}^{\nu}}{x_{23}^2}\right) - \frac{x_{12}^2}{x_{13}^2 x_{23}^2}\frac{\eta^{\mu\nu}}{d} \ . \tag{3.4.5}$$

The OPE coefficient $C_{\mathcal{O}\mathcal{O}T}$ is fixed by the Ward identity

$$C_{\mathcal{O}\mathcal{O}T} = -\Delta_{\mathcal{O}} \frac{\Gamma(d/2)d}{2\pi^{d/2}(d-1)} \ . \tag{3.4.6}$$

We therefore wish to compute

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = -\frac{iC_{\mathcal{O}\mathcal{O}T}B^{d-2}}{\rho}\int_{-\infty}^{\infty} d\tilde{u} \int_{\vec{x}_3^2 \le \rho} d^{d-2}\vec{x}_3 \int d\tau d^{d-2}\vec{y} \frac{(\rho - \vec{x}_3^2) I_{uu}}{x_{23}^{d-2} x_{12}^{2\Delta_{\mathcal{O}}+2-d} x_{13}^{d-2}} \tag{3.4.7}$$

in the large B limit, where points x_1, x_2 and x_3 are given by (3.4.3).

Performing the $\tilde{u}$-integral:

In our coordinates, we find that the factors in the denominator have the form

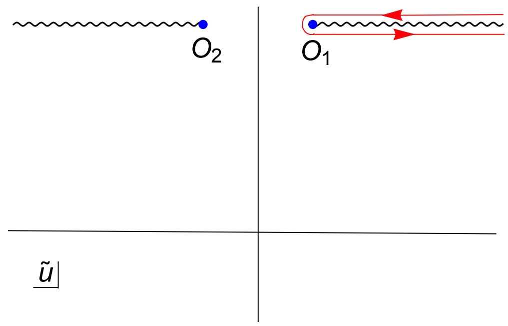
$$x_{13}^2 = c_1\tilde{u} + c_2, \qquad x_{23}^2 = c_3\tilde{u} + c_4 \ , \tag{3.4.8}$$

where c_i's are $\tilde{u}$-independent complex constants and the numerator will in general be a finite degree polynomial $P(\tilde{u})$ in $\tilde{u}$. If we perform the $\tilde{u}$-integral with the usual $i\epsilon$-prescription, then the $\tilde{u}$-contour does not enclose any poles (or branch cuts) and the integral vanishes. Instead, we need to follow a prescription similar to the prescription of [62] to obtain the operator ordering of (3.4.3). Whenever the holographic null energy operator appears inside a correlator, we define the $\tilde{u}$-integral with the $\tilde{u}$-contour such that the $\tilde{u}$-integral in (3.4.3) is determined by the residue at the pole due to the operator O_1 (in the presence of branch cuts the integral is determined by the integral of the discontinuity across the branch cut due to the operator O_1). This contour can be motivated in many different ways. In equation (3.4.3), both the stress tensor and the operator O_2 are smeared over some region. To give a physical interpretation of the contour, consider centers of these smeared operators:

$$\int_{-\infty}^{\infty} d\tilde{u} \langle \overline{\varepsilon_1}.O_1(iB, -\delta) T_{uu}(\tilde{u}, 0) \varepsilon_2.O_2(iB, \delta) \rangle \ . \tag{3.4.9}$$

In general this $\tilde{u}$-integral has branch cut singularities at $u = iB \pm \delta$. And the above contour is equivalent to the prescription of analytic continuation of [62]. Another way to understand this choice of contour is to perform a $\pi/2$ rotation in the Euclidean $\tau - x^1$ plane and start with (3.2.5) instead of (3.4.3). Now if we consider the centers of the smeared operators, the choice of contour for $\tilde{u}$-integral is obvious.

To summarize, effectively the $\tilde{u}$-integral in (3.4.3) is given by the contour:



(3.4.10)

Let us now use this contour to perform integrals of the form:

$$\int_{-\infty}^{+\infty} d\tilde{u} \frac{P(\tilde{u})}{(c_1\tilde{u}+c_2)^{a_1}(c_3\tilde{u}+c_4)^{a_2}} \equiv \int_\gamma d\tilde{u} \frac{P(\tilde{u})}{(c_1\tilde{u}+c_2)^{a_1}(c_3\tilde{u}+c_4)^{a_2}} \ , \tag{3.4.11}$$

where $P(u)$ is a polynomial in u. These integrals can be easily evaluated by using the identity

$$\int_\gamma d\tilde{u} \frac{1}{(\tilde{u}+c_2)^{p_1}(c_4-\tilde{u})^{p_2}} = \frac{2\pi i}{(c_4+c_2)^{p_1+p_2-1}} \frac{\Gamma(p_1+p_2-1)}{\Gamma(p_1)\Gamma(p_2)} \ , \tag{3.4.12}$$

where, p_1 and p_2 are positive numbers with $p_1+p_2>1$. So, now performing the $\tilde{u}$-integral and taking the large-B limit we find, [18]

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = \frac{\pi 2^{2d-3}\Gamma(d+1)C_{\mathcal{O}\mathcal{O}T}}{\rho\Gamma\left(\frac{d}{2}+1\right)^2} \int_{\vec{x}_3^2\leq\rho} d^{d-2}\vec{x}_3 \int d^{d-1}\vec{\mathbf{p}} \frac{(\vec{\mathbf{x}}_3\cdot\vec{\mathbf{x}}_3-\rho)(1-\vec{\mathbf{x}}_3\cdot\vec{\mathbf{x}}_3)^{1-d}}{(\vec{\mathbf{p}}^2+\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{1-d/2+\Delta_{\mathcal{O}}}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{d-1}}, \tag{3.4.13}$$

where we have made a change of variables from $(\tau, \vec{y})$ to $\vec{\mathbf{p}}$ and defined the following $(d-1)$-dimensional vectors running over time and $D-2$ transverse coordinates

[18] Naively it seems that $\vec{\mathbf{p}}$ integral is divergent near $\vec{\mathbf{p}} \to 0$. However, $\vec{\mathbf{p}}$ is a complex valued vector and the integration region is shifted in the imaginary direction. In practice this means that the integration must be performed by analytic continuation using appropriate choice of contours to ensure convergence as described in appendix A.8.

$(\tau, \vec{y})$

$$
\begin{aligned}
\vec{\mathbf{k}} &= (\tau, \vec{y}) \ , \\
\vec{\mathbf{p}} &= \vec{\mathbf{k}} - \frac{\vec{\mathbf{L}}}{2} \ , \\
\vec{\mathbf{L}} &= \frac{8\delta\vec{\mathbf{x}}_3 + 4i\delta(\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{x}}_3 + 1)\vec{\mathbf{T}}}{\vec{\mathbf{x}}_3 \cdot \vec{\mathbf{x}}_3 - 1} \ , \\
\vec{\mathbf{T}} &= (1, \vec{0}) \ , \\
\vec{\mathbf{x}}_3 &= (0, \vec{x}_3) \ .
\end{aligned}
\tag{3.4.14}
$$

Before we proceed, let us note that if one starts with (3.2.5) instead of (3.4.3), the $\tilde{u}$-integral should be performed in a similar way. After taking the large-B limit, one ends up with exactly (3.4.13) and hence the rest of the calculation is identical.

Performing the $\vec{\mathbf{p}}$-integral:

It turns out that even in the most general correlation function, the smearing integrals reduce to the form

$$
\int d^{d-1}\vec{\mathbf{p}} \frac{\prod_i \vec{\mathbf{p}}.\vec{v}_i}{(\vec{\mathbf{p}}^2 + \vec{\mathbf{p}} \cdot \vec{\mathbf{L}})^{p_1} (\vec{\mathbf{p}} \cdot \vec{\mathbf{L}})^{p_2}}, \tag{3.4.15}
$$

where $\vec{v}_i$ are constant vectors. These integrals have closed form expressions in the most general case and the relevant results are summarized in appendix A.8. In this example, performing the smearing integrals yields[19]

$$
\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = \frac{\pi^{d/2}\Gamma\left(\frac{d+1}{2}\right) 2^{2\left(d-\Delta_{\mathcal{O}}-\frac{3}{2}\right)}\Gamma\left(-\frac{d}{2}+\Delta_{\mathcal{O}}+\frac{3}{2}\right) C_{\mathcal{O}\mathcal{O}T}}{\rho\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\Delta_{\mathcal{O}}+1\right)} \int_{\vec{x}_3^2 \le \rho} d^{d-2}\vec{x}_3 \frac{\vec{x}_3 \cdot \vec{x}_3 - \rho}{(1 - \vec{x}_3 \cdot \vec{x}_3)^{d-1}}. \tag{3.4.16}
$$

[19] From now on we set $\delta = 1$ for simplicity. In the final expression one can restore δ back by dimensional analysis.

Performing the $\vec{x}_3$ integral:

The most general integrals of the kind that appeared in our last expression, after going to the radial coordinate, can be done using

$$\int_0^{\sqrt{\rho}} dx \left(1-x^2\right)^a x^b \left(x^2-\rho\right)^c = \frac{\rho^{\frac{b+1}{2}}\Gamma\left(\frac{b+1}{2}\right)(-\rho)^c\Gamma(c+1)\,{}_2F_1\left(-a,\frac{b+1}{2};\frac{b+3}{2}+c;\rho\right)}{2\Gamma\left(\frac{b+3}{2}+c\right)}\ , \tag{3.4.17}$$

where, $b, c > -1$ and $0 < \rho < 1$. Using this identity we finally obtain

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = -\frac{\pi^{d-1}\rho^{\frac{d}{2}-1}C_{\mathcal{O}\mathcal{O}T}\Gamma\left(\frac{d+1}{2}\right)\,{}_2F_1\left(\frac{d}{2}-1,d-1;\frac{d}{2}+1;\rho\right)4^{d-\Delta_{\mathcal{O}}-\frac{3}{2}}\Gamma\left(-\frac{d}{2}+\Delta_{\mathcal{O}}+\frac{3}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(\Delta_{\mathcal{O}}+1\right)}. \tag{3.4.18}$$

For scalars, the causality condition $\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) \geq 0$ is already satisfied because of the Ward identity. Note that $\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho)$ satisfies the relation (3.3.9) which is the first check of the UV/IR connection.[20] As described in the previous section the lightcone limit is obtained by taking $\rho \to 0$:

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = -\frac{\pi^{d-1}\rho^{\frac{d}{2}-1}\Gamma\left(\frac{d+1}{2}\right)4^{d-\Delta_{\mathcal{O}}-\frac{3}{2}}C_{\mathcal{O}\mathcal{O}T}\Gamma\left(-\frac{d}{2}+\Delta_{\mathcal{O}}+\frac{3}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(\Delta_{\mathcal{O}}+1\right)} + \mathcal{O}(\rho^{d/2}). \tag{3.4.19}$$

The "bulk-point" limit[21] is obtained by taking the limit $\rho \to 1$ and in $D \geq 4$, we obtain:

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = -\frac{d\pi^{d-1}\Gamma\left(\frac{d+1}{2}\right)4^{d-\Delta_{\mathcal{O}}-\frac{5}{2}}C_{\mathcal{O}\mathcal{O}T}\Gamma\left(-\frac{d}{2}+\Delta_{\mathcal{O}}+\frac{3}{2}\right)}{(d-3)\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(\Delta_{\mathcal{O}}+1\right)(1-\rho)^{d-3}} + \mathcal{O}(1-\rho)^{4-d}. \tag{3.4.20}$$

[20] Let us recall that for scalars $\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = f_{\mathcal{O}\mathcal{O}}(\rho)$.

[21] This bulk point limit is similar to the bulk point limit studied in [92], however, it is not exactly the same. Our bulk point limit is in fact the limit discussed in [9] in the context of causality.

In $D=3$, there is a logarithmic divergence in the limit $\rho \to 1$

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) = \frac{4^{\frac{5}{2}-\Delta_{\mathcal{O}}}\pi C_{\mathcal{O}\mathcal{O}T}}{3\Delta_{\mathcal{O}}} \ln(1-\rho) + \mathcal{O}(1)\ . \tag{3.4.21}$$

3.4.2 Spinning operators

It was shown in [6, 60] that the most general 3-point functions of symmetric traceless spinning operators in a CFT can be written as a sum over certain elementary spinning structures:

$$\langle \Phi_1 \Phi_2 \Phi_3 \rangle = \sum_{\{n_{23},n_{13},n_{12}\}} C^{\Phi_1\Phi_2\Phi_3}_{n_{23},n_{13},n_{12}} \frac{V_{1,23}^{\ell_1-n_{12}-n_{13}} V_{2,13}^{\ell_2-n_{12}-n_{23}} V_{3,12}^{\ell_3-n_{13}-n_{23}} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{(x_{12}^2)^{\frac{1}{2}(h_1+h_2-h_3)} (x_{13}^2)^{\frac{1}{2}(h_1+h_3-h_2)} (x_{23}^2)^{\frac{1}{2}(h_2+h_3-h_1)}}, \tag{3.4.22}$$

where $C^{\Phi_1\Phi_2\Phi_3}_{n_{23},n_{13},n_{12}}$ are constant coefficients and $h_i \equiv \Delta_i + \ell_i$. The structures are given by

$$H_{ij} \equiv x_{ij}^2 \varepsilon_i \cdot \varepsilon_j - 2(x_{ij}\cdot\varepsilon_i)(x_{ij}\cdot\varepsilon_j), \qquad V_{i,jk} \equiv \frac{x_{ij}^2 x_{ik}\cdot\varepsilon_i - x_{ik}^2 x_{ij}\cdot\varepsilon_i}{x_{jk}^2}\ , \tag{3.4.23}$$

where, $x_{ij}^\mu = (x_i - x_j)^\mu$ and ε_i is a null polarization vector contracted with spinning indices of Φ_i in the following way:

$$(\varepsilon^\mu \varepsilon^\nu \cdots)\, \Phi_{\mu\nu\cdots} \equiv \varepsilon.\Phi\ . \tag{3.4.24}$$

For a traceless symmetric tensor, one can easily convert the null polarization $\varepsilon^\mu\varepsilon^\nu\cdots$ into an arbitrary polarization tensor $\varepsilon^{\mu\nu\cdots}$ by using projection operators [60].

The sum in (A.13.4) is over all triplets of non-negative integers $\{n_{12}, n_{13}, n_{23}\}$ satisfying

$$\ell_1 - n_{12} - n_{13} \geq 0\ , \qquad \ell_2 - n_{12} - n_{23} \geq 0\ , \qquad \ell_3 - n_{13} - n_{23} \geq 0\ . \tag{3.4.25}$$

For a general correlation function, the coefficients $C^{\Phi_1\Phi_2\Phi_3}_{n_{23},n_{13},n_{12}}$ are all independent parameters, however, imposing conservation equations or Ward identities will impose relations amongst these coefficients.

From equation (A.13.4), we see that the most general integrals in $\mathbf{E}_{O_1O_2}(\rho)$ are of the form

$$\int d\tilde{u}\int_{\vec{x}_3^2\leq\rho} d^{d-2}\vec{x}_3\int d\tau d^{d-2}\vec{y}\frac{(\rho-\vec{x}_3^2)V^{a_1}_{1,23}V^{a_2}_{2,13}V^{a_3}_{3,12}H^{b_1}_{12}H^{b_2}_{13}H^{b_3}_{23}}{(x_{12}^2)^{\frac{1}{2}(h_1+h_2-h_3)}(x_{13}^2)^{\frac{1}{2}(h_1+h_3-h_2)}(x_{23}^2)^{\frac{1}{2}(h_2+h_3-h_1)}}\ , \tag{3.4.26}$$

where the exponents in the numerator, a_i and b_i, are positive integers. Polarizations are given by (in $D\geq 4$)[22]

$$\varepsilon_1^\mu=(1,\xi_1,\vec{\varepsilon}_{1,\perp})\ ,\qquad \varepsilon_2^\mu=(1,\xi_2,\vec{\varepsilon}_{2,\perp})\ ,\qquad \varepsilon_3^\mu=\frac{1}{2}(1,-1,\vec{0})\ , \tag{3.4.27}$$

with $\xi_{1,2}=\pm1$ and $\vec{\varepsilon}_{1,\perp}{}^2=\vec{\varepsilon}_{2,\perp}{}^2=0$.

Angular integrals:

In the case where the external operators are non-scalars, similar to (3.4.14) we also need to introduce $(d-1)$-dimensional vectors made out of the polarization vectors ε_1^μ, ε_2^μ:

$$\vec{\boldsymbol{\epsilon}}_{1,\perp}=(0,\vec{\varepsilon}_{1,\perp}),\qquad \vec{\boldsymbol{\epsilon}}_{2,\perp}=(0,\vec{\varepsilon}_{2,\perp}). \tag{3.4.28}$$

Now after $\vec{\mathbf{p}}$-integrals, we will have to perform angular integrals for $\vec{x}_3$ which is of the form

$$\int_{\mathbb{S}^{d-3}} d\hat{\Omega}(\vec{\varepsilon}_{1,\perp}\cdot\vec{x}_3)^n(\vec{\varepsilon}_{2,\perp}\cdot\vec{x}_3)^m=\frac{\pi^{\frac{d-2}{2}}2^{1-n}|\vec{x}_3|^{2n}\Gamma(n+1)(\vec{\varepsilon}_{1,\perp}\cdot\vec{\varepsilon}_{2,\perp})^n\delta_{m,n}}{\Gamma\left(\frac{d-2}{2}+n\right)}, \tag{3.4.29}$$

[22]We will treat the $D=3$ case separately.

where $d\hat{\Omega}$ is the standard measure on $\mathbb{S}^{d-3}$ and we have used the fact that $\vec{\varepsilon}_{2,\perp}{}^2 = \vec{\varepsilon}_{1,\perp}{}^2 = 0$. Rest of the computation is identical to the scalar case and can be efficiently automated in Mathematica.

3.5 Bounds on $\langle TTT\rangle$, $\langle JJT\rangle$, and $\langle \mathcal{O}_{\ell=1,2}\mathcal{O}_{\ell=1,2}T\rangle$

In this section, we will use the methods described above to derive constraints in $D \geq 4$. These constraints encompasses, and generalizes, the constraints obtained in [9, 25, 26, 62, 66] by studying various four-point functions in holographic CFTs. Note that the $D = 3$ case is more subtle which we will discuss in a separate section.

3.5.1 $\langle JJT\rangle$

We start with $\mathbf{E}_{JJ}$ where J is a spin-1 conserved current. The $\langle JJT\rangle$ three-point function is given in Appendix A.6.1. Following our formalism, the leading term in the limit $\rho \to 1$ is given by

$$\mathbf{E}_{JJ}(\rho) \sim \frac{-2^{-d-2}\pi^{d-\frac{1}{2}}\left(d-2\left(\lambda^2+1\right)\right)\Gamma\left(\frac{d-1}{2}\right)(4n_f-n_s)}{\Gamma\left(\left(\frac{d}{2}+1\right)^2\Gamma\left(\frac{d}{2}\right)(1-\rho)^{d-1}\right.} + \mathcal{O}\left(\frac{1}{(1-\rho)^{2-d}}\right) \tag{3.5.1}$$

up to some positive overall coefficient. Our choice of polarizations is given in equation (4.3.14) with $\varepsilon_2^\mu = \overline{\varepsilon}_1^\mu$ and

$$\lambda^2 = \frac{1}{2}\vec{\overline{\varepsilon}}_{2,\perp}\cdot\vec{\varepsilon}_{2,\perp} \geq 0\ . \tag{3.5.2}$$

As shown in the Appendix A.9, given our choice of polarization, different powers of λ^2 correspond to independent spinning structures and decomposition of $SO(d-$

1, 1) to representations under $SO(d-2)$. Therefore, positivity of $\mathbf{E}_{JJ}$ implies that coefficients of each powers of λ^2 must be positive. Hence, from equation (3.5.1) we obtain

$$n_s = 4n_f + \mathcal{O}\left(\frac{c_J}{\Delta_{\text{gap}}^2}\right) = \frac{d(d-2)}{S_d(d-1)}c_J + \mathcal{O}\left(\frac{c_J}{\Delta_{\text{gap}}^2}\right) , \tag{3.5.3}$$

where, in the last equation we have used the Ward identity (A.6.6). The Δ_{gap} correction in the above equation is computed following (3.2.29). All subleading contributions to (3.5.1) are proportional to $n_s\,(2\lambda^2+1)$, a manifestly positive quantity. Therefore, subleading terms of $\mathbf{E}_{JJ}(\rho)$ do not lead to new constraints. Furthermore, it is obvious from (3.5.3) that the three-point function $\langle JJT\rangle$ is completely determined by the $\langle JJ\rangle$ two-point function. In fact, this is a general feature of CFTs with a large central charge and a large gap.

After imposing the constraint (3.5.2), we can compute $f_{JJ}(\rho)$:

$$f_{\varepsilon_1\cdot J\,\varepsilon_2\cdot J}(\rho) = \frac{2^{-d}\pi^{d-\frac{1}{2}}\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}+1\right)^2\Gamma\left(\frac{d}{2}\right)} n_s\left(1+\vec{\varepsilon}_{1,\perp}\cdot\vec{\varepsilon}_{2,\perp}\right)\rho^{\frac{d}{2}-1}\,{}_2F_1\left(\frac{d}{2}-1, d-1; \frac{d}{2}+1; \rho\right) \tag{3.5.4}$$

which is consistent with the equation (3.3.9).

In dual gravity language, the three-point function $\langle JJT\rangle$ arises from the following action of a massless gauge field

$$\int d^{d+1}x\sqrt{-g}\ \left[-F_{\mu\nu}F^{\mu\nu} + \alpha_{AAh}W_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\right], \tag{3.5.5}$$

where, W is the Weyl tensor[23]. The coefficient α_{AAh} can be written in terms of n_s and n_f:

$$\alpha_{AAh} \sim \frac{n_s - 4n_f}{n_s + 4(d-2)n_f} \sim \frac{1}{\Delta_{\text{gap}}^2} . \tag{3.5.6}$$

[23] The Weyl tensor is given by $W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{D-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{(D-1)(D-2)}Rg_{\mu[\rho}g_{\sigma]\nu}$, where $D = d+1$.

Hence, α_{AAh} should be suppressed by the scale of new physics. The power dependence of the suppression $\alpha_{AAh} \sim \frac{1}{\Delta_{\rm gap}^2}$ agrees with the result obtained from causality of the effective field theory in the bulk [5].[24]

3.5.2 $\langle TTT \rangle$

Let us now consider $\mathbf{E}_{TT}(\rho)$ where $\langle TTT \rangle$ three-point function is given in Appendix A.6.2. Following our formalism, the leading term in the limit $\rho \to 1$ is given by

$$\mathbf{E}_{TT}(\rho) \sim \frac{(-1)^d 4^{1-d} \pi^d \Gamma(d) \left(-8d\lambda^2 + (d-2)d + 8\lambda^4\right)}{(d-2)\Gamma\left(\frac{d}{2}+1\right)^2 \Gamma\left(\frac{d}{2}+2\right)\Gamma\left(\frac{d}{2}\right)(1-\rho)^{d+1}} \times \left((d-2)d^2(4\tilde{n}_f - \tilde{n}_s) - 64(d-3)\tilde{n}_v\right) + \mathcal{O}\left(\frac{1}{(1-\rho)^d}\right) \tag{3.5.7}$$

up to some overall positive coefficient. Polarizations are given by equation (4.3.14) with $\varepsilon_2^\mu = \bar{\varepsilon}_1^\mu$ and λ is defined in equation (3.5.2). Positivity of $\mathbf{E}_{TT}$ for all powers of λ demands that we must have

$$\tilde{n}_v = \frac{(d-2)d^2(4\tilde{n}_f - \tilde{n}_s)}{64(d-3)} + \mathcal{O}\left(\frac{c_T}{\Delta_{\rm gap}^4}\right). \tag{3.5.8}$$

After imposing this condition, the next leading term becomes

$$\mathbf{E}_{TT}(\rho) \sim \frac{(-1)^{d-1} 2^{1-d}(d+1)\pi^{d-\frac{1}{2}}\Gamma\left(\frac{d-3}{2}\right)}{(d-1)\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}\right)^2(1-\rho)^{d-1}} \left(d^2 - 4(d-1)\lambda^4 + 2(d-3)(d-1)\lambda^2 - 5d + 6\right) \times \left(2(d-1)\tilde{n}_f - (d-2)\tilde{n}_s\right) + \mathcal{O}\left(\frac{1}{(1-\rho)^d}\right). \tag{3.5.9}$$

[24] Here we are assuming $R_{AdS} = 1$.

Positivity then implies

$$\begin{aligned}
\tilde{n}_f &= \frac{(d-2)\tilde{n}_s}{2(d-1)} + \mathcal{O}\left(\frac{c_T}{\Delta_{\text{gap}}^2}\right) \ , \\
\tilde{n}_v &= \frac{(d-2)d^2\tilde{n}_s}{64(d-1)} + \mathcal{O}\left(\frac{c_T}{\Delta_{\text{gap}}^2}\right) \ , \\
\tilde{n}_s &= \frac{c_T(d-1)}{32(d-2)(d+1)S_d} + \mathcal{O}\left(\frac{c_T}{\Delta_{\text{gap}}^2}\right) \ ,
\end{aligned} \tag{3.5.10}$$

where, we have also used the Ward identity (A.6.12) to derive the last equation. After imposing these constraints, the positivity of $\tilde{n}_s$ guarantees that the rest of the terms are always positive and hence no new constraints are obtained from subleading terms. Note that the three-point function $\langle TTT \rangle$ is completely determined by the $\langle TT \rangle$ two-point function. Furthermore, we can now compute our $f_{\varepsilon_1 \cdot T\, \varepsilon_2 \cdot T}(\rho)$ function

$$\begin{aligned}
f_{\varepsilon_1 \cdot T\, \varepsilon_2 \cdot T}(\rho) = &\frac{((d-1)(\vec{\varepsilon}_{1,\perp} \cdot \vec{\varepsilon}_{2,\perp})^2 + 2(d-1)\vec{\varepsilon}_{1,\perp} \cdot \vec{\varepsilon}_{2,\perp} + d - 2)}{(d-1)^2 \Gamma\left(\frac{d}{2}-1\right)\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}\right)} \\
&\times\ \tilde{n}_s \pi^{d-1/2} 2^{5-d} \Gamma\left(\frac{d+3}{2}\right) \rho^{\frac{d}{2}-1} {}_2F_1\left(\frac{d}{2}-1, d-1; \frac{d}{2}+1; \rho\right)
\end{aligned} \tag{3.5.11}$$

which is in agreement with the relation (3.3.9) indicating that the Regge OPE of smeared operators is indeed universal.

On the gravity side, this constrains higher derivative correction terms in the pure gravity action that contribute to three point interactions of gravitons. These higher derivative correction terms can be parametrized as

$$S = M_{pl}^{d-1} \int d^{d+1}x \sqrt{g}\ \left[R - 2\Lambda + \alpha_2 W_{\mu\nu\alpha\beta} W^{\mu\nu\alpha\beta} + \alpha_4 W_{\mu\nu\alpha\beta} W^{\mu\nu\rho\sigma} W_{\rho\sigma}{}^{\alpha\beta}\right] . \tag{3.5.12}$$

Note that in case of vacuum AdS, Weyl tensor vanishes. Other terms are encoding the most general form of three-point interaction for gravitons. Coupling constants

α_2 and α_4 are related to the coefficients $\tilde{n}_s$, $\tilde{n}_f$ and $\tilde{n}_v$. In particular, constraints (3.5.10) translate into $\alpha_2 \lesssim \frac{1}{\Delta_{\rm gap}^2}, \alpha_4 \lesssim \frac{1}{\Delta_{\rm gap}^4}$ (for $D \geq 4$) which is in agreement with the expectation from bulk causality [5].

Conformal trace anomaly in 6d

In $D = 4$, the causality constraints (3.5.10) can be rewritten as a statement about central charges: $\frac{|a-c|}{c} \lesssim 1/\Delta_{\rm gap}^2$. There is a similar relation between trace anomaly coefficients in $D = 6$. In particular, the conformal trace anomaly in $D = 6$ can be written as [93–96]

$$\langle T^\mu_\mu \rangle = 2a_6 E_6 + c_1 I_1 + c_2 I_2 + c_3 I_3 \tag{3.5.13}$$

up to total derivative terms which can be removed by adding finite and covariant counter-terms in the effective action. In equation (3.5.13), a_6, c_1, c_2, c_3 are 6d central charges and

$$\begin{aligned} I_1 =& W_{\gamma\alpha\beta\delta} W^{\alpha\mu\nu\beta} W_{\mu\ \ \nu}^{\ \gamma\delta} \ , \\ I_2 =& W_{\alpha\beta}^{\ \ \ \gamma\delta} W_{\gamma\delta}^{\ \ \ \mu\nu} W_{\mu\nu}^{\ \ \ \alpha\beta} \ , \\ I_3 =& W_{\alpha\gamma\delta\mu} \left(\nabla^2 \delta^\alpha_\beta + 4 R^\alpha_\beta - \frac{6}{5} R \delta^\alpha_\beta \right) W^{\beta\gamma\delta\mu} \ , \\ E_6 =& \frac{1}{384(2\pi)^3} \delta^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}_{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6} R^{\nu_1\nu_2}_{\ \ \ \ \ \mu_1\mu_2} R^{\nu_3\nu_4}_{\ \ \ \ \ \mu_3\mu_4} R^{\nu_5\nu_6}_{\ \ \ \ \ \mu_5\mu_6} \ . \end{aligned} \tag{3.5.14}$$

The a_6 coefficient can be determined only from the stress tensor four-point function and hence (3.5.10) does not constrain a_6. However, c_1, c_2, c_3 are related to the stress tensor three-point function and hence constraints (3.5.10) can be translated into constraints on central charges. In particular, using the result of [97] for Einstein gravity, we can easily show that

$$\frac{c_1}{c_3} = -12 + \mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}^2}\right) , \qquad \frac{c_2}{c_3} = -3 + \mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}^2}\right) . \tag{3.5.15}$$

Note that the relations between c_1, c_2, c_3 are exactly what is expected for $(2,0)$ supersymmetric theories. For these theories, invariants I_1, I_2, I_3 can be combined into a single super-invariant [98–100] which leads to the relation: $c_1 = 4c_2 = -12c_3$ [90]. This relation between c_1, c_2, c_3 was first derived in [95] for the free $(2,0)$ tensor multiplet. On the other hand, the same relation also holds for strongly coupled theories with a supergravity dual [7].

3.5.3 $\langle \mathcal{O}_{\ell=1}\mathcal{O}_{\ell=1}T\rangle$

Now we derive bounds on non-conserved spin-1 operators. The three point function $\langle \mathcal{O}_{\ell=1}\mathcal{O}_{\ell=1}T\rangle$ has five OPE constants $\{C_{0,0,0}, C_{0,0,1}, C_{0,1,0}, C_{1,0,0}, C_{1,1,0}\}$. Imposing permutation symmetry and conservation equation reduces this number to three independent coefficients. The leading contribution in the limit $\rho \to 1$ is

$$\begin{aligned}\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) \sim & -\frac{\pi^{d-1}(1-\rho)^{1-d}(d-\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp-2)\Gamma\left(\frac{d+1}{2}\right)2^{2d-2\Delta_\mathcal{O}-3}\Gamma\left(-\frac{d}{2}+\Delta_\mathcal{O}+\frac{3}{2}\right)}{(d-1)d\sigma\Gamma\left(\frac{d}{2}\right)^2\left(d-2\left(\Delta_\mathcal{O}+1\right)\right)\Gamma\left(\Delta_\mathcal{O}+2\right)} \\ & \times\Big(C_{1,1,0}\left(d^2+d\left(2\Delta_\mathcal{O}\left(\Delta_\mathcal{O}+1\right)-1\right)-2\left(\Delta_\mathcal{O}\left(\Delta_\mathcal{O}+3\right)+1\right)\right) \\ & -2(d-1)C_{0,0,1}+C_{0,1,0}\left(4\Delta_\mathcal{O}+2\right)\Big) ,\end{aligned} \tag{3.5.16}$$

where we have used the polarizations $\varepsilon = (1, \xi, \vec{\varepsilon}_\perp)$ and $\bar{\varepsilon} = (-1, -\xi, \bar{\vec{\varepsilon}}_\perp)$ with $\xi = \pm 1$. Imposing positivity on the coefficients of powers of $\vec{\varepsilon}_\perp \cdot \bar{\vec{\varepsilon}}_\perp$ we find

$$C_{1,1,0} = \frac{2(d-1)C_{0,0,1} - 2C_{0,1,0}\left(2\Delta_O+1\right)}{d^2+d\left(2\Delta_O\left(\Delta_O+1\right)-1\right)-2\left(\Delta_O\left(\Delta_O+3\right)+1\right)} + \mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}^2}\right) . \tag{3.5.17}$$

After imposing this condition the next leading term is

$$\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) \sim \frac{\pi^{d-1}4^{d-\Delta_O-1}\xi\left(2C_{0,0,1}\left((d-1)\Delta_O-1\right)+C_{0,1,0}\left(d-2\Delta_O^2\right)\right)}{d\Gamma\left(\frac{d}{2}\right)^2\left(d^2+d\left(2\Delta_O\left(\Delta_O+1\right)-1\right)-2\left(\Delta_O\left(\Delta_O+3\right)+1\right)\right)\Gamma\left(\Delta_O+1\right)} \times \frac{1}{(1-\rho)^{d-2}}\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(-\frac{d}{2}+\Delta_O+\frac{3}{2}\right) \ . \tag{3.5.18}$$

As described previously, the above expression must be positive for $\xi=\pm1$ resulting in

$$C_{0,1,0}=\frac{2C_{0,0,1}\left(d\Delta_O-\Delta_O-1\right)}{2\Delta_O^2-d}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ . \tag{3.5.19}$$

After imposing the condition, the resulting expression

$$-\frac{d\pi^{d-1}C_{0,0,1}\Gamma\left(\frac{d+1}{2}\right)2^{2d-2\Delta_O-5}\Gamma\left(-\frac{d}{2}+\Delta_O+\frac{3}{2}\right)\left(d-\Delta_O(\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp+2)+\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp\right)}{(1-\rho)^{d-3}(d-3)\Gamma\left(\frac{d}{2}+1\right)^2\left(d-2\Delta_O^2\right)\Gamma\left(\Delta_O\right)} + \mathcal{O}\left(\frac{1}{(1-\rho)^{d-4}}\right) \tag{3.5.20}$$

has only one independent coefficient $C_{0,0,1}$ and is positive if and only if $C_{0,0,1}<0$.

Finally, imposing causality constraints and conservation equation result in the following relations

$$\begin{aligned} C_{0,0,1}&=\frac{C_{0,0,0}\left(d-2\Delta_O^2\right)}{d^2-4d\Delta_O+4\Delta_O}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\ C_{0,1,0}=C_{1,0,0}&=-\frac{2C_{0,0,0}\left(d\Delta_O-\Delta_O-1\right)}{d^2-4d\Delta_O+4\Delta_O}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\ C_{1,1,0}&=\frac{2C_{0,0,0}}{d^2-4d\Delta_O+4\Delta_O}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \end{aligned} \tag{3.5.21}$$

and hence there is only one independent coefficient which is related to the two-point function $\langle\mathcal{O}_{\ell=1}\mathcal{O}_{\ell=1}\rangle$ by the Ward identity. Similarly, we can show that after imposing the causality constraints

$$f_{\varepsilon_1\cdot\mathcal{O}\,\varepsilon_2\cdot\mathcal{O}}(\rho)=-\frac{\Gamma\left(\frac{d+1}{2}\right)2^{2d-2\Delta_\mathcal{O}-2}C_{0,0,1}\Gamma\left(-\frac{d}{2}+\Delta_\mathcal{O}+\frac{3}{2}\right)\left(-d+2\Delta_\mathcal{O}+\left(\Delta_\mathcal{O}-1\right)\vec{\varepsilon}_{1,\perp}\cdot\vec{\varepsilon}_{2,\perp}\right)}{\Gamma\left(\frac{d}{2}+1\right)^2\left(2\Delta_\mathcal{O}^2-d\right)\Gamma\left(\Delta_\mathcal{O}\right)} \times\pi^{d-1}\rho^{\frac{d}{2}-1}{}_2F_1\left(\frac{d}{2}-1,d-1;\frac{d}{2}+1;\rho\right) \tag{3.5.22}$$

which is consistent with the equation (3.3.9).

In the gravity side, the causality constraints imply that the action for a massive spin-1 field in the bulk must have the form

$$\int d^{d+1}x\sqrt{-g}\ \left[-F_{\mu\nu}F^{\mu\nu}+m^2A_\mu A^\mu+\cdots\right]\ , \tag{3.5.23}$$

where dots represent higher derivative terms (for example $W_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}$, $A_\mu A_\nu R^{\mu\nu}$) which must be suppressed by scale of new physics in the gravity side.

3.5.4 $\langle \mathcal{O}_{\ell=2}\mathcal{O}_{\ell=2}T\rangle$

Similarly, we can find bounds for non-conserved spin-2 operator $\mathcal{O}_{\ell=2}$. For simplicity we quote the results in 4 dimensions but it can be easily generalized in general D. We assume that the three-point function $\langle \mathcal{O}_{\ell=2}TT\rangle$ vanishes so that it does not appear as an exchange operator in the Regge four-point function. But the three-point function $\langle \mathcal{O}_{\ell=2}\mathcal{O}_{\ell=2}T\rangle$ is non-vanishing and to begin with it has 11 coupling constants. Permutation symmetry and conservation equation ensure that only 6 of these coefficients are independent. Furthermore, causality demands that only one of these coefficient can be independent. In particular, the leading contribution in the limit $\rho\to 1$ is given by

$$\begin{aligned}\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho)\sim &-\frac{\pi^4 2^{3-4\Delta_\mathcal{O}}\left(\Delta_\mathcal{O}+1\right)\left(\Delta_\mathcal{O}+2\right)\Gamma\left(2\Delta_\mathcal{O}-2\right)\left((\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp-8)\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp+4\right)}{(1-\rho)^5\Gamma\left(\Delta_\mathcal{O}+3\right)^2}\\ &\times\left(2C_{0,1,0}\left(2\Delta_\mathcal{O}+3\right)\left(\Delta_\mathcal{O}\left(\Delta_\mathcal{O}+3\right)+6\right)-24C_{0,1,1}\left(2\Delta_\mathcal{O}+3\right)+72C_{0,0,2}\right.\\ &+36\left(2C_{0,2,0}+C_{1,1,0}-6C_{1,1,1}\right)+\Delta_\mathcal{O}\left(-2C_{0,2,0}\left(\Delta_\mathcal{O}+1\right)\left(\Delta_\mathcal{O}\left(3\Delta_\mathcal{O}+19\right)+18\right)\right.\\ &\left.\left.+C_{1,1,0}\left(\Delta_\mathcal{O}\left(\Delta_\mathcal{O}\left(3\Delta_\mathcal{O}+14\right)+43\right)+60\right)-24C_{1,1,1}\left(3\Delta_\mathcal{O}+7\right)\right)\right)\ .\end{aligned} \tag{3.5.24}$$

Following the same procedure as for spin 1 and including conservation conditions we find

$$
\begin{aligned}
C_{0,0,1} &= \frac{C_{0,0,0}\left(3\Delta_{\mathcal{O}}^3-2\Delta_{\mathcal{O}}^2-15\Delta_{\mathcal{O}}+18\right)}{4\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{0,0,2} &= \frac{C_{0,0,0}\left(\Delta_{\mathcal{O}}^4-5\Delta_{\mathcal{O}}^2+8\right)}{16\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{0,1,0} = C_{1,0,0} &= \frac{C_{0,0,0}\left(6\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}-1\right)}{2\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{0,1,1} = C_{1,0,1} &= \frac{C_{0,0,0}\Delta_{\mathcal{O}}\left(3\Delta_{\mathcal{O}}^2+4\Delta_{\mathcal{O}}-15\right)}{8\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{0,2,0} = C_{2,0,0} &= \frac{C_{0,0,0}\left(3\Delta_{\mathcal{O}}^2-2\right)}{4\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{1,1,0} &= \frac{C_{0,0,0}\left(3\Delta_{\mathcal{O}}^2+1\right)}{2\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ , \\
C_{1,1,1} &= \frac{C_{0,0,0}\Delta_{\mathcal{O}}^2}{2\left(3\Delta_{\mathcal{O}}^2-9\Delta_{\mathcal{O}}+7\right)}+\mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}}\right) \ .
\end{aligned}
\tag{3.5.25}
$$

Imposing these conditions we find that the subleading term

$$
\begin{aligned}
\mathbf{E}_{\mathcal{O}\mathcal{O}}(\rho) \sim& -\frac{3\pi^4 4^{-2\Delta_{\mathcal{O}}-1}\left(\Delta_{\mathcal{O}}-1\right)\Gamma\left(2\Delta_{\mathcal{O}}-1\right)C_{0,0,0}}{(1-\rho)\left(3\left(\Delta_{\mathcal{O}}-3\right)\Delta_{\mathcal{O}}+7\right)\Gamma\left(\Delta_{\mathcal{O}}\right)^2} \\
&\times \Big(4\left(\Delta_{\mathcal{O}}-3\right)\left(\Delta_{\mathcal{O}}-2\right)+\left(\Delta_{\mathcal{O}}-1\right)\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp\left(\Delta_{\mathcal{O}}(\vec{\varepsilon}_\perp\cdot\bar{\vec{\varepsilon}}_\perp+4)-8\right)\Big) \ ,
\end{aligned}
\tag{3.5.26}
$$

is determined up to one independent coefficient $C_{0,0,0} < 0$. This coefficient is related to the coefficient that appears in the two-point function $\langle\mathcal{O}_{\ell=2}\mathcal{O}_{\ell=2}\rangle$ by the Ward identity. Furthermore, after imposing all of the constraints we find that

$$
\begin{aligned}
f_{\varepsilon_1\cdot\mathcal{O}\,\varepsilon_2\cdot\mathcal{O}}(\rho) =& -\frac{3\pi^4 4^{-2\Delta_{\mathcal{O}}-\frac{1}{2}}\left(\Delta_{\mathcal{O}}-1\right)\Gamma\left(2\Delta_{\mathcal{O}}-1\right)\rho}{\left(3\left(\Delta_{\mathcal{O}}-3\right)\Delta_{\mathcal{O}}+7\right)\Gamma\left(\Delta_{\mathcal{O}}\right)^2(1-\rho)}C_{0,0,0} \\
&\times (4(\Delta_{\mathcal{O}}-3)(\Delta_{\mathcal{O}}-2)+(\Delta_{\mathcal{O}}-1)\vec{\varepsilon}_{1,\perp}\cdot\vec{\varepsilon}_{2,\perp}(4\Delta_{\mathcal{O}}+\Delta_{\mathcal{O}}\vec{\varepsilon}_{1,\perp}\cdot\vec{\varepsilon}_{2,\perp}-8))
\end{aligned}
\tag{3.5.27}
$$

which is consistent with the universality of the Regge OPE of smeared operators.

In the gravity dual description, there are also 6 possible types of vertices appearing in the on-shell three-point function of 2 massive spin-2 particles with a single graviton. The CFT result shows that the final answer is fixed up to a constant which is in agreement with the gravity result. Furthermore, requiring causality in the bulk [5, 101] dictates that the three-point function is determined up to a constant corresponding to the minimal coupling between massive spin 2 fields and a graviton. The vertex has the following form

$$((\epsilon_2 \cdot \epsilon_3)(\epsilon_1 \cdot p_2) + (\epsilon_1 \cdot \epsilon_3)(\epsilon_2 \cdot p_3) + (\epsilon_1 \cdot \epsilon_2)(\epsilon_3 \cdot p_1))^2 , \tag{3.5.28}$$

where the momenta are denoted by p_1, p_2, p_3, satisfying conservation and on-shell conditions: $p_1^\mu + p_2^\mu + p_3^\mu = 0, p_1^2 = -m^2, p_2^2 = -m^2, p_3^2 = 0$ and ϵ_i denote polarization tensors. For a more complete analysis of vertices and bulk dual, see [101, 102].

3.6 Bounds from interference effect

In this section, we will leverage the holographic null energy condition to derive bounds on the off-diagonal matrix elements of the operator $\mathbf{E}_r$. To this end we will consider superposition states created by smeared local operators:

$$-i \lim_{B\to\infty} \langle (\overline{\varepsilon_1.O_1(B)} + \overline{\varepsilon_2.O_2(B)})\mathbf{E}_{r=\sqrt{\rho}B}(\varepsilon_1.O_1(B) + \varepsilon_2.O_2(B))\rangle \geq 0 \tag{3.6.1}$$

where O_1 and O_2 are arbitrary operators with or without spin ($\ell_1, \ell_2 \leq 2$). This inequality can be expressed as semi-definiteness of the following matrix

$$\begin{pmatrix} \mathbf{E}_{O_1O_1}(\rho) & \mathbf{E}_{O_1O_2}(\rho) \\ \mathbf{E}_{O_1O_2}(\rho)^* & \mathbf{E}_{O_2O_2}(\rho) \end{pmatrix} \succeq 0 \ , \tag{3.6.2}$$

where, we are using the notation (3.4.1). The above condition can also be restated in the following form

$$|\mathbf{E}_{O_1O_2}(\rho)|^2 \leq \mathbf{E}_{O_1O_1}(\rho)\mathbf{E}_{O_2O_2}(\rho) \ , \qquad 0 < \rho < 1 \ . \tag{3.6.3}$$

This is very similar to the interference effects in conformal collider experiment as studied in [65]. In particular, in the limit $\rho \to 0$, the above relation is equivalent to the interference effects of [65]. However, we are interested in the limit $\rho \to 1$ in which the above inequality imposes stronger constraints on three-point functions $\langle O_1 O_2 T\rangle$. These interference bounds are exactly the same as the bounds obtained in [66] by studying mixed system of four-point functions in the Regge limit in holographic CFTs.

As shown in the previous section, in $D \geq 4$ after imposing positivity of $\mathbf{E}_{O_1O_1}(\rho)$ we have

$$\mathbf{E}_{O_1O_1}(\rho) \sim \mathcal{O}(1-\rho)^{3-d} \ . \tag{3.6.4}$$

Similarly,

$$\mathbf{E}_{O_2O_2}(\rho) \sim \mathcal{O}(1-\rho)^{3-d} \ . \tag{3.6.5}$$

Therefore, $\mathbf{E}_{O_1O_2}(\rho)$ can not grow faster than $\mathcal{O}(1-\rho)^{3-d}$ in the limit $\rho \to 1$, or else causality will be violated. However, just from dimensional argument one can show that, in general

$$\mathbf{E}_{O_1O_2}(\rho) \sim \frac{1}{(1-\rho)^{-3+d+\ell_1+\ell_2}} \sum_{n=0,1,\cdots} c_n (1-\rho)^n \tag{3.6.6}$$

and hence

$$c_n = \mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}^{\ell_1+\ell_2-n}}\right) \ , \qquad n = 0, 1, \cdots, \ell_1+\ell_2-1 \ . \tag{3.6.7}$$

Whereas, $c_{\ell_1+\ell_2}$ is constrained by (4.2.38).

The causality conditions (3.6.7) are too constraining. In fact, from simple counting, one can argue that constraints (3.6.7) require all three-point functions of the form $\langle T O_1 O_2\rangle$ to vanish. Constraints (3.6.7) lead to at least $\ell_1 + \ell_2$ linear algebraic equations among the OPE coefficients of $\langle TO_1O_2\rangle$. However, for low spin operators ($\ell \leq 2$), the number of independent OPE coefficients of $\langle TO_1O_2\rangle$ is always less than $\ell_1+\ell_2$. This immediately suggests

$$\langle TO_1O_2\rangle = 0 \ , \tag{3.6.8}$$

when O_1 and O_2 are different operators. Explicit computations, as we will show next, confirm that this is indeed true when O_1, O_2 are single trace primary operators. All our results are consistent with the interference bounds obtained in [66] by using the conformal Regge theory.

Bound on $\langle TT\psi\rangle$

As an example, we will obtain bounds on the OPE coefficient $C_{TT\psi}$ of $\langle TT\psi\rangle$ in $D\geq 4$ where ψ is a light scalar operator. The polarization of T is still given by $(1,\xi,\vec{\varepsilon}_\perp)$. Now, from (4.2.37) we have

$$\begin{pmatrix} \mathcal{O}(1-\rho)^{3-d} & c_0(1-\rho)^{1-d}+\mathcal{O}(1-\rho)^{2-d} \\ c_0(1-\rho)^{1-d}+\mathcal{O}(1-\rho)^{2-d} & \mathcal{O}(1-\rho)^{3-d} \end{pmatrix} \succeq 0. \tag{3.6.9}$$

Positivity of the eigenvalues of this matrix implies

$$\begin{aligned} c_0 &\sim \frac{\pi^{d-1}\Gamma\left(\frac{d}{2}-\frac{1}{2}\right)2^{d-\Delta_\psi-5}e^{-\frac{1}{2}i\pi\left(d+\Delta_\psi\right)}\Gamma\left(\frac{\Delta_\psi}{2}+\frac{3}{2}\right)}{(1-\rho)^{d-1}\Gamma\left(\frac{\Delta_\psi}{2}+2\right)\Gamma\left(d-\frac{\Delta_\psi}{2}\right)\Gamma\left(\frac{d}{2}+\frac{\Delta_\psi}{2}+1\right)}C_{TT\psi} \\ &\sim \mathcal{O}\left(\frac{1}{\Delta_{\text{gap}}^2}\right) \end{aligned} \tag{3.6.10}$$

and hence

$$\langle TT\psi\rangle \lesssim \mathcal{O}\left(\frac{\sqrt{c_T}}{\Delta_{\text{gap}}^2}\right) \tag{3.6.11}$$

for all values of Δ_ψ for which the coefficient in front of $C_{TT\psi}$ does not vanish. Note that the coefficient in front of $C_{TT\psi}$ vanishes when $\Delta_\psi = 2d+2n$ which is consistent with the fact that there are double trace stress tensor operators $[TT]_{\ell=0,n}$ which have spin 0. This agrees with the result obtained in [66].

In the dual gravity picture, $\langle TT\psi\rangle$ vanishes for a minimally coupled scalar field in AdS. However, in the bulk we can write higher derivative interactions between a scalar and two gravitons which give rise to $\langle TT\psi\rangle$ three-point function. In particular, let us consider the bulk action

$$S = \frac{M_{Pl}^{d-1}}{2}\int d^{d+1}x\sqrt{g}\left[(\nabla\Psi)^2 - m^2\Psi^2\right] + M_{Pl}^{d-1}\alpha_{\Psi hh}\int d^{d+1}x\sqrt{g}\,\Psi W^2\ . \tag{3.6.12}$$

In $D \geq 4$, the scalar-graviton-graviton vertex of the above action represents the most general bulk interaction which gives rise to the OPE coefficient $C_{TT\psi}$ [65]:

$$\frac{C_{TT\psi}}{\sqrt{c_T}} = \alpha_{\Psi hh}\frac{8\pi^{d/2}(d-1)\sqrt{2d}}{\sqrt{d+1}\,\Gamma(d/2)\sqrt{f(\Delta_\psi)}} \tag{3.6.13}$$

where, the function $f(\Delta)$ is given in [65]. Hence, $\alpha_{\Psi hh}$ should be suppressed by the scale of new physics. In particular, the causality constraint (3.6.11) translates into $\alpha_{\Psi hh} \lesssim \frac{1}{\Delta_{\rm gap}^2}$.[25] Of course, this is stronger than the constraint obtained in [65]. In [65], constraints were obtained by considering interference effects in general CFTs. However, as shown in (3.6.7), interference effects from the holographic null energy condition lead to stronger constraints.

[25]Note that $\Gamma\left(d-\frac{\Delta_\psi}{2}\right)$ in equation (3.6.10) is canceled by $\sqrt{f(\Delta_\psi)}$ and hence the constraint $\alpha_{\Psi hh} \lesssim \frac{1}{\Delta_{\rm gap}^2}$ is valid for any mass of the scalar field Ψ.

Bound on $\langle TT\mathcal{O}_{\ell=2}\rangle$

Let us now obtain bounds on the three-point function $\langle TT\mathcal{O}_{\ell=2}\rangle$. This case is more subtle because a nonzero $\langle TT\mathcal{O}_{\ell=2}\rangle$ implies that the operator $\mathcal{O}_{\ell=2}$ will contribute to a four-point function in the Regge limit as an exchange operator. So, if $\langle TT\mathcal{O}_{\ell=2}\rangle \neq 0$, the holographic null energy condition is no longer true. However, simplification emerges if we assume that there is at least one heavy scalar in the theory ψ_H for which $\langle \psi_H\psi_H\mathcal{O}_{\ell=2}\rangle = 0$. In this case, we can start with the operator ψ_H in (3.2.13) and derive the holographic null energy condition even in the presence of $\mathcal{O}_{\ell=2}$. So, with this additional assumption, we can calculate $\mathbf{E}_{T\mathcal{O}_{\ell=2}}(\rho)$ which is a straight forward generalization of the scalar case. Furthermore, the interference condition (3.6.7) again leads to

$$\langle TT\mathcal{O}_{\ell=2}\rangle \lesssim \mathcal{O}\left(\frac{\sqrt{c_T}}{\Delta_{\text{gap}}}\right) \ . \tag{3.6.14}$$

Let us note that the above bound is not applicable when the dimension of $\mathcal{O}_{\ell=2}$ satisfies: $\Delta_{\mathcal{O}_{\ell=2}} = 2d + 2 + 2n$. This is consistent with the fact that there are double trace stress tensor operators $[TT]_{\ell=2,n}$ with spin 2. With this caveat, we conclude that the presence of a single heavy scalar operator ψ_H guarantees that the three-point function $\langle TT\mathcal{O}_{\ell=2}\rangle$ is suppressed by the gap for all single trace $\mathcal{O}_{\ell=2}$. An immediate consequence is that the operator $\mathcal{O}_{\ell=2}$ can not contribute as an exchange operator to four-point functions $\langle TT\mathcal{O}\mathcal{O}\rangle$ in the Regge limit for any $\mathcal{O}$.

Before we proceed, let us also note that we expect that the same conclusion is true even without the presence of ψ_H. We believe causality of the four-point function $\langle TTTT\rangle$, requires that $\langle TT\mathcal{O}_{\ell=2}\rangle$ must be suppressed by the gap for all single trace $\mathcal{O}_{\ell=2}$. However, a detailed analysis requires the computation of

$\langle TTTT\rangle$ using the conformal Regge theory which we will not attempt in this chapter.

3.7 Constraints on CFTs in $D = 3$

In this section, we will use the holographic null energy condition in $(2 + 1)$-dimensions to constrain various three-point functions of $(2+1)$-dimensional CFTs. Three-dimensional CFTs are special because of the presence of various parity odd structures. However, we again show that CFTs in $D = 3$ with large central charge and a large gap exhibit universal, gravity-like behavior. Furthermore, holography enables us to translate the CFT bounds in to constraints on $(3 + 1)$-dimensional gravitational interactions. This, as we will discuss in the next section, has important consequences in cosmology.

There is another aspect of $D = 3$ which is different from the higher dimensional case. For $D \geq 4$, we have seen that holographic dual of a bulk derivative is $1/\Delta_{\text{gap}}$. This observation is consistent with the proposal of [66]. However, we will show that in $D = 3$, this simple relationship between bulk derivative and Δ_{gap} has a logarithmic violation.

3.7.1 $\langle TTT\rangle$

In (2+1) dimensions, $\langle TTT\rangle$ has three tensor structures: two parity even structures with coefficients $\tilde{n}_s$ and $\tilde{n}_f$, and one parity odd structure with coefficient $\tilde{n}_{\text{odd}}$ (see appendix A.7). We start with the holographic null energy condition (3.2.21) with

O being the stress-tensor T. In the limit $\rho \to 1$, the leading contribution to $\mathbf{E}_{TT}(\rho)$ goes as $\frac{1}{(1-\rho)^4}$, the coefficient of which should always be positive. In particular,

$$\mathbf{E}_{TT}(\rho)|_{\text{Even}} \sim \frac{32\pi\left(4\tilde{n}_f - \tilde{n}_s\right)}{5(1-\rho)^4}\left(e_0^2\left(\bar{e}_0^2+\bar{e}_2^2\right) - 4e_2e_0\bar{e}_0\bar{e}_2 + e_2^2\left(\bar{e}_0^2+\bar{e}_2^2\right)\right),$$
$$\mathbf{E}_{TT}(\rho)|_{\text{Odd}} \sim \frac{8\pi^2 i\tilde{n}_{\text{odd}}\left(e_0^2\bar{e}_0\bar{e}_2 - e_2e_0\left(\bar{e}_0^2+\bar{e}_2^2\right) + e_2^2\bar{e}_0\bar{e}_2\right)}{15(1-\rho)^4}, \tag{3.7.1}$$

where we have defined

$$\varepsilon = (e_0, e_1, e_2)\ , \qquad \bar{\varepsilon} = (\bar{e}_0, \bar{e}_1, \bar{e}_2). \tag{3.7.2}$$

The total expression can be conveniently written as

$$\mathbf{E}_{TT}(\rho) \sim \frac{8\pi\left(-i\pi\tilde{n}_{\text{odd}}AB + 12\left(4\tilde{n}_f - \tilde{n}_s\right)\left(A^2+B^2\right)\right)}{15(1-\rho)^4}, \tag{3.7.3}$$

where

$$A \equiv |e_0|^2 - |e_2|^2\ , \qquad B \equiv e_2e_0^* - e_0e_2^*\ . \tag{3.7.4}$$

To find constraints on the coefficients, we first choose

$$\varepsilon = (i, \sqrt{-2}, 1) \Rightarrow (4\tilde{n}_f - \tilde{n}_s) \geq 0,$$
$$\varepsilon = (0, i, 1) \Rightarrow (4\tilde{n}_f - \tilde{n}_s) \leq 0, \tag{3.7.5}$$

implying that $\tilde{n}_s = 4\tilde{n}_f$. Imposing this condition we find constraints on the parity odd structure by considering

$$\varepsilon = (1+i, \sqrt{-1+2i}, 1) \Rightarrow \tilde{n}_{\text{odd}} \geq 0,$$
$$\varepsilon = (-1+i, \sqrt{-1-2i}, -1) \Rightarrow \tilde{n}_{\text{odd}} \leq 0, \tag{3.7.6}$$

implying that we must have $\tilde{n}_{\text{odd}} = 0$ to satisfy positivity. Furthermore, after imposing these constraints, one can check that $f_{\varepsilon_0\cdot T\,\varepsilon_1\cdot T}(\rho)$ is still given by the equation (3.5.11) with $D = 3$.

Let us now estimate the size of the corrections to the above constrains if we include higher spin operators with large scaling dimensions, but not large enough to compete with the c_T expansion. We can repeat the argument of section 3.2.3 for $D = 3$, yielding

$$\frac{|\tilde{n}_s - 4\tilde{n}_f|}{c_T} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^4} , \qquad \frac{|\tilde{n}_{\text{odd}}|}{c_T} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^4} . \tag{3.7.7}$$

On the gravity side, similar to the higher dimensional case, this constrains higher derivative correction terms in the pure gravity action that contribute to three point interactions of gravitons. However, in $(3+1)-$dimensional gravity there are certain crucial differences. First, the four-derivative terms do not contribute to $\langle TTT \rangle$. Second, in $(3+1)-$dimensional gravity, there is a parity odd higher derivative term which gives rise to $\tilde{n}_{\text{odd}}$. In particular, the higher derivative correction terms can be parametrized as

$$S = M_{Pl}^2 \int d^4x \sqrt{g} \, \left[R - 2\Lambda + \alpha_4 W_{\mu\nu\alpha\beta} W^{\mu\nu\rho\sigma} W_{\rho\sigma}{}^{\alpha\beta} + \tilde{\alpha}_4 \tilde{W}_{\mu\nu\alpha\beta} W^{\mu\nu\rho\sigma} W_{\rho\sigma}{}^{\alpha\beta} \right] \ , \tag{3.7.8}$$

where, $\tilde{W}_{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} W^{\rho\sigma}{}_{\alpha\beta}$. Coupling constants α_4 and $\tilde{\alpha}_4$ are related to the coefficients $\tilde{n}_s - 4\tilde{n}_f$ and $\tilde{n}_{\text{odd}}$ respectively.[26] Hence, causality constraints translate into $\alpha_4 \sim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^4}, \tilde{\alpha}_4 \sim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^4}$. It was proposed in [66] that holographic dual of a bulk derivative is $1/\Delta_{\text{gap}}$. However, as we see here, for $(3+1)-$dimensional gravity, there is a logarithmic violation.

[26] In $(3+1)-$dimensional gravity, one can also have another parity violating term in the four-derivative level: $\int d^4x\sqrt{g}\tilde{W}W$ which is a total derivative. However, this term contributes a non-trivial parity violating contact term to the two-point function $\langle TT \rangle$ [103].

3.7.2 $\langle JJT \rangle$

Similarly, in $(2+1)$ dimensions $\langle JJT \rangle$ has parity even and odd structures (see appendix A.7) with the leading terms in the limit $\rho \to 1$ given by

$$\mathbf{E}_{JJ}(\rho)|_{\text{Even}} \sim -\frac{\pi\,(e_0\bar{e}_0 - e_2\bar{e}_2)\,(4n_f - n_s)}{9(1-\rho)^2}\ ,$$
$$\mathbf{E}_{JJ}(\rho)|_{\text{Odd}} \sim \frac{2i\pi^2 n_{\text{odd}}\,(e_2\bar{e}_0 - e_0\bar{e}_2)}{3(1-\rho)^2}\ . \qquad (3.7.9)$$

Positivity of $\mathbf{E}_{JJ}(\rho)$ implies the following conditions on the coefficients

$$\frac{|n_s - 4n_f|}{c_J} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2}\ , \qquad \frac{|n_{\text{odd}}|}{c_J} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2}\ . \qquad (3.7.10)$$

After imposing these constraints, one can easily check that our conjectured relation (3.3.9) is satisfied.

The three-point function $\langle JJT \rangle$, in dual gravity language, arises from the following 4d-action

$$\int d^4x\sqrt{-g}\ \left[-F_{\mu\nu}F^{\mu\nu} + \alpha_{AAh}W_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + \tilde{\alpha}_{AAh}\tilde{W}_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}\right], \qquad (3.7.11)$$

where, coefficients α_{AAh} and $\tilde{\alpha}_{AAh}$ can be written in terms of n_s, n_f and n_{odd}:

$$\alpha_{AAh} \sim \frac{n_s - 4n_f}{n_s + 4n_f} \sim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2}\ , \qquad \tilde{\alpha}_{AAh} \sim \frac{n_{\text{odd}}}{n_s + 4n_f} \sim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2}\ . \qquad (3.7.12)$$

Appearance of $\ln \Delta_{\text{gap}}$ again indicates that the simple relationship between bulk derivative and Δ_{gap} has a logarithmic violation in 3d CFT.

3.7.3 $\langle TT\psi \rangle$

Let us now discuss the three-point function $\langle TT\psi \rangle$ in $D = 3$. The analysis is identical to the derivation of causality constraints for $\langle TT\psi \rangle$ in higher dimension using interference effects. So, we will not show the full calculation, instead we only point out the key differences. In $D = 3$, conformal invariance also allows for a parity odd structure and the full correlator consists of two structures

$$\langle TT\psi \rangle = \langle TT\psi \rangle_{\text{Even}} + \langle TT\psi \rangle_{\text{Odd}} \tag{3.7.13}$$

with OPE coefficients $C^{\text{Even}}_{TT\psi}$ and $C^{\text{Odd}}_{TT\psi}$ respectively [65]. First, we derive causality constraints on the three-point function $\langle TTT \rangle$ which leads to (3.7.7). After imposing these constraints, in the limit $\rho \to 1$, $\mathbf{E}_{TT}(\rho) \sim \ln(1-\rho)$. On the other hand, in the limit $\rho \to 1$, for both even and odd structures $\mathbf{E}_{T\psi}(\rho) \sim \frac{1}{(1-\rho)^2}$. Hence, the interference bound (4.2.38) dictates that both $C^{\text{Even}}_{TT\psi}$ and $C^{\text{Odd}}_{TT\psi}$ should be suppressed by Δ_{gap}:

$$\frac{C^{\text{Even}}_{TT\psi}}{\sqrt{c_T}} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta^2_{\text{gap}}} \ , \qquad \frac{C^{\text{Odd}}_{TT\psi}}{\sqrt{c_T}} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta^2_{\text{gap}}} \ . \tag{3.7.14}$$

Similarly, in the bulk there are two possible vertices between a scalar and two gravitons, one parity even and one parity odd. These interactions can be parametrized as

$$S = M^2_{Pl}\alpha_{\Psi hh} \int d^4x\sqrt{g}\, \Psi W^2 + M^2_{Pl}\tilde{\alpha}_{\Psi hh} \int d^4x\sqrt{g}\, \Psi \tilde{W} W \ . \tag{3.7.15}$$

These interactions were first constrained by Cordova, Maldacena, and Turiaci in [65]. Using the averaged null energy condition they showed that in generic CFTs in $D = 3$, interference effects impose constraints on the OPE coefficients $C^{\text{Even}}_{TT\psi}$ and $C^{\text{Odd}}_{TT\psi}$. These general bounds can be translated into bounds on gravitational interactions [65]

$$\sqrt{\alpha^2_{\Psi hh} + \tilde{\alpha}^2_{\Psi hh}} \leq \frac{1}{12\sqrt{2}} \ . \tag{3.7.16}$$

However, it is expected that the holographic null energy condition leads to stronger constraints on $\alpha_{\Psi hh}, \tilde{\alpha}_{\Psi hh}$. In particular, bounds (3.7.14) are equivalent to

$$\alpha_{\Psi hh} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2} \ , \qquad \tilde{\alpha}_{\Psi hh} \lesssim \frac{\ln \Delta_{\text{gap}}}{\Delta_{\text{gap}}^2} \ . \tag{3.7.17}$$

In the following section, we will use these constraints to impose bounds on inflationary observables.

3.8 Constraining inflationary observables

In the last section, we showed that $(2+1)-$dimensional CFTs with large central charge and a sparse spectrum, irrespective of their microscopic details, admit universal holographic dual description at low energies. This connection provides us with a tool to constrain gravitational interactions in $(3+1)-$dimensions. This has an immediate application in inflation. The period of inflation is an exponential expansion of the universe that powered the epoch of the big bang. On one hand, inflation naturally explains why our universe is flat and homogeneous on the large scale. On the other hand, quantum fluctuations produced during inflation are responsible for the temperature fluctuations of cosmic microwave background (CMB) and the large-scale structures of the universe.

The simplest model of inflation consists of a real scalar field minimally coupled to Einstein gravity. In general, there can be higher derivative interactions which can contribute to various inflationary observables. Therefore, constraints obtained in the previous section can impose bounds on such observables (for example chiral gravity waves, tensor-to-scalar ratio etc.). However, there is a caveat. All of the constraints on gravitational interactions obtained in this chapter, strictly speaking, are valid in AdS. Following the philosophy of [5, 65], we simply assume that the same constraints are also valid in de Sitter after we make the substitution $R_{AdS} \to 1/H$, where H is the Hubble scale associated with inflation. This is a reasonable assumption but it would be important to have a robust derivation of these de Sitter constraints.

3.8.1 Chiral gravity waves

Chiral gravity waves [104,105] can be produced during inflation from a parity odd higher derivative interaction in the action

$$M_{Pl}^2 \int d^4x \sqrt{g}\, f_o(\Psi)\tilde{W}W \ , \qquad \Psi = \frac{\phi}{M_{Pl}} \ , \tag{3.8.1}$$

where ϕ is the inflaton field. In the presence of this term in the action, two-point functions of tensor modes with left handed and right handed circular polarizations are not the same. The asymmetry A measures the difference between left and right handed polarizations and it is determined by the above parity odd interaction [65]

$$A \equiv \frac{\langle h_L h_L\rangle - \langle h_R h_R\rangle}{\langle h_L h_L\rangle + \langle h_R h_R\rangle} = \pm 4\pi\sqrt{2\epsilon}\,\tilde{\alpha}_{\Psi hh} H^2 \ , \tag{3.8.2}$$

where, ϵ is one of the slow-roll parameters of inflation. In the above expression, we have used the fact $\tilde{\alpha}_{\Psi hh} = \frac{\partial f_o(\Psi)}{\partial \Psi}$. So, constraint (3.7.17) strongly suggests that the asymmetry parameter A must be suppressed by the scale of new physics M:[27]

$$|A| \lesssim 4\pi\sqrt{2\epsilon}\left(\frac{H^2}{M^2}\right)\ln\left(\frac{M}{H}\right) \ . \tag{3.8.3}$$

First of all, note that this is stronger than the bound obtained in [65]. Secondly, if the asymmetry parameter A is measured (or in other words it is found to be at least a few percent), then the new physics scale must be $M \sim H$. This scenario

[27]We have identified $\Delta_{\text{gap}} = M/H$.

necessarily requires the presence of an infinite tower of new particles with spins $\ell > 2$ and masses $\sim H$. Therefore, any detection of this parameter in future experiments will serve as an evidence in favor of string theory (or something very similar) with the string scale comparable to the Hubble scale and a very small coupling.

3.8.2 Tensor-to-scalar ratio

Similarly, one can obtain a bound on the ratio r of the amplitudes of tensor fluctuations and scalar fluctuations. In a single-field inflation without any higher derivative couplings, the tensor-to-scalar ratio r obeys a consistency condition [106]: $r = -8n_t$, where n_t is the tensor spectral index. In the presence of the higher derivative interaction

$$M_{Pl}^2 \int d^4x \sqrt{g}\, f_e(\Psi) W^2 \ , \tag{3.8.4}$$

the consistency condition is violated [107]. In particular, one can show that [65]

$$-\frac{n_t}{r} = \frac{1}{8} \pm \frac{H^2}{\sqrt{2\epsilon}} \alpha_{\Psi hh} \ , \qquad \alpha_{\Psi hh} = \frac{\partial f_e(\Psi)}{\partial \Psi} \ . \tag{3.8.5}$$

In the above expression we have assumed that the inflaton field has only a canonical kinetic term with two-derivatives.[28] So far, this is exactly the same as the discussion of [65]. But we now derive a stronger bound by using constraint (3.7.17) which suggests that the violation of the consistency relation must be suppressed by M

$$\left|\frac{n_t}{r} + \frac{1}{8}\right| \lesssim \frac{1}{\sqrt{2\epsilon}} \left(\frac{H^2}{M^2}\right) \ln\left(\frac{M}{H}\right) \ . \tag{3.8.6}$$

[28] In other words, the speed of sound for the inflaton field is 1.

3.8.3 Graviton non-gaussanity

Let us now consider non-gaussanity of primordial gravitational waves produced during inflation. In Einstein gravity, the three-point function of tensor perturbation goes as

$$\langle hhh \rangle_E \sim \frac{H^4}{M_{Pl}^4} \ . \tag{3.8.7}$$

The graviton three-point function (parity preserving part) can also get contributions from W^3 term in the gravity action (3.7.8). As shown in [5], the contribution from this interaction must be suppressed by the scale of new physics:

$$\frac{\langle hhh \rangle_{W^3}}{\langle hhh \rangle_E} \sim \alpha_4 H^4 \sim \left(\frac{H^4}{M^4}\right) \ln\left(\frac{M}{H}\right) \ . \tag{3.8.8}$$

Hence, any significant deviation from the Einstein gravity result requires the presence of an infinite tower of new particles with spins $\ell > 2$ and masses $\sim H$ [5].

The advantage of studying any parity violating effects during inflation is that these contributions are exactly zero for Einstein gravity. Hence, any detection of parity violation will be a signature of new physics at the Hubble scale. The gravity action in general can have a parity odd term $\tilde{W}W^2$ which is also controlled by the same scale M. This term contributes to the parity odd part of graviton three-point function [108–110]. In particular,

$$\langle h_L h_L h_L \rangle - \langle h_R h_R h_R \rangle \sim \epsilon \left(\frac{H^4}{M_{Pl}^4}\right) \tilde{\alpha}_4 H^4 \ . \tag{3.8.9}$$

Therefore, causality requires that

$$\frac{\langle h_L h_L h_L \rangle - \langle h_R h_R h_R \rangle}{\langle hhh \rangle} \sim \epsilon \left(\frac{H^4}{M^4}\right) \ln\left(\frac{M}{H}\right) \ . \tag{3.8.10}$$

This parity violating graviton non-gaussanity will have signatures in the CMB. For example, CMB three-points correlators $\langle TEB \rangle$, $\langle EEB \rangle$, $\langle TTB \rangle$ become nonzero in the presence of the parity violating graviton non-gaussanity. However, one disadvantage of studying the parity violating graviton non-gaussanity is that this contribution is exactly zero in pure de Sitter [108,110]. Hence, for slow-roll inflation this effect is suppressed by the slow-roll parameter ϵ.

We should also note that terms like $f_e(\phi)W^2$ or $f_o(\phi)\tilde{W}W$ in the effective action can also contribute to the graviton three-point function. Both these contributions depend on the details of the inflationary scenario and they can dominate over the contributions from W^3 and $\tilde{W}W^2$ [110, 111]. However, contributions of $f_e(\phi)W^2$ and $f_o(\phi)\tilde{W}W$ to the graviton three-point function are proportional to $\sqrt{\epsilon}f_e'(\phi)$ and $\sqrt{\epsilon}f_o'(\phi)$ respectively which are bounded by causality as well. So, if these terms are present in the effective action, their contributions to the non-gaussanity of primordial gravitational waves should also be suppressed by M but with a different power

$$\frac{\langle hhh\rangle_{f_e(\phi)W^2,\ f_o(\phi)\tilde{W}W}}{\langle hhh\rangle_E} \sim \sqrt{\epsilon}\left(\frac{H^2}{M^2}\right)\ln\left(\frac{M}{H}\right) \ . \tag{3.8.11}$$

3.9 Discussion

In this chapter, we analyzed the implications of causality of correlation functions on CFT data in theories with large c_T and sparse higher spin spectrum. This was accomplished by developing a new formalism that can be interpreted as a collider type experiment in the CFT, set up in such a way to probe scattering processes deep in the bulk interior of the corresponding holographic dual theory. In doing so we consider the *holographic null energy operator*, $\mathbf{E}_r$ which is a positive operator in a certain subspace of the total CFT Hilbert space. This subspace is spanned by states constructed by acting local operators, smeared with Gaussian wave-packets, on the CFT vacuum. Positivity of this operator was then used to impose bounds on the CFT data.

Other representations

It is worth mentioning that the formalism presented here can easily be adopted to compute the contribution of the holographic null-energy operator to the four-point function of external operators in arbitrary representation including spinors or non-symmetric traceless representations. The only modification required is to compute three-point functions of these operators with the stress-tensor whose form is fixed by conformal symmetry.

Furthermore, with slight modification one may compute the contribution of single-trace exchanged operators other than the stress-tensor. More specifically in [62] it was shown that in the Regge limit ($v \to 0$ with uv held fixed) the contribution of a spinning operator X (with spin ℓ and dimension Δ_X) to the OPE can be written as

$$\begin{aligned}\frac{\psi(u,v)\psi(-u,-v)|_X}{\langle\psi(u,v)\psi(-u,-v)\rangle} &= \pi^{\frac{1-d}{2}} 2^{\Delta_X} \frac{\Gamma(\frac{\Delta_X+\ell+1}{2})}{\Gamma(\frac{\Delta_X+\ell}{2})} \frac{\Gamma(\Delta_X - d/2+1)}{\Gamma(\Delta_X - d+2)} \frac{C_{\psi\psi X}}{C_X} \\ &\times \frac{(-uv)^{\frac{d-\ell-\Delta_X}{2}}}{u^{1-\ell}} \int_{-\infty}^{+\infty} d\tilde{u} \int_{\vec{x}^2\leq -uv} d^{d-2}\vec{x}(-uv-\vec{x}^2)^{\Delta_X-d+1} X_{uu\cdots u}(\tilde{u},0,i\vec{x})\end{aligned} \quad . \quad (3.9.1)$$

This OPE is valid as long as it is evaluated in a correlation function where all other operator insertions are held fixed as we take the Regge limit. However, the chaos bound suggests that this contribution does not necessarily dominate in the Regge limit in holographic CFTs.

Non-conserved spin-2 exchange

As previously mentioned, one caveat to our computation is the possibility of competition between the contributions of non-conserved spin-2 operators with the stress-tensor in the Regge limit. However, using the OPE described above it is possible to explicitly compute the contribution of such an operator to the Regge OPE. Including the contribution of a single non-conserved spin-2 exchange, we find bounds on the OPE coefficients of the stress-tensor as well as the non-conserved spin-2 operator. We expect that some version of the experiment described above, should reproduce the constraints found in [102] which resulted from performing a scattering experiment in the bulk. We leave explicit confirmation of this claim to future explorations.

Regge OPE of single trace operators

The operator product expansion of smeared primary operators in the Regge limit, as discussed in section 3.3, is universal. When O_1 and O_2 are different operators, the identity piece in the OPE (3.3.10) does not contribute. Moreover, if O_1 and O_2 are single trace operators, then interference effects imply that $\langle TO_1O_2\rangle = 0$. So, for these operators, even the coefficient of the shockwave operator in (3.3.10) vanishes. Hence, for non-identical single trace primary operators the OPE

$$\Psi^*[O_1]\Psi[O_2] = 0 + \cdots , \tag{3.9.2}$$

where, dots represent terms which are suppressed by either the large gap limit or the large c_T limit or the Regge limit.

Higher spin ANEC

Although not pursued in detail here, by taking the lightcone limit of (3.9.1), the same formalism developed here can be used to compute the contribution of the ANEC operator to correlation functions. Furthermore, this formalism can be easily extended to study the higher spin ANEC [10] which says

$$\int du X_{uu\cdots u} \geq 0 \ , \tag{3.9.3}$$

where, X is the lowest dimension operator with even spin ($\ell \geq 2$). Positivity of these operators holds in the more general class of theories including non-holographic CFTs. A systematic exploration of bounds derived from the positivity of these operators is left to future work.

OPE of spinning operators

It would be interesting to derive the stress tensor contribution to the OPE of spinning operators both in the Regge and the lightcone limits. Using this OPE, an argument similar to the ones used in this chapter would lead to new positive spinning null energy conditions. These positivity conditions both conceptually as well as technically, will have important implications. For instance, this will allow us to derive new constraints in a more systematic way. Moreover, based on the analogous constraints obtained in the bulk [5], we expect these positive operators to play an important role in closing the gap in ruling out non-conserved spin-2 exchanges.

CHAPTER 4

A BOUND ON MASSIVE HIGHER SPIN PARTICLES

Abstract[1]

According to common lore, massive elementary higher spin particles lead to inconsistencies when coupled to gravity. However, this scenario was not completely ruled out by previous arguments. In this chapter, we show that in a theory where the low energy dynamics of the gravitons are governed by the Einstein-Hilbert action, any finite number of massive elementary particles with spin more than two cannot interact with gravitons, even classically, in a way that preserves causality. This is achieved in flat spacetime by studying eikonal scattering of higher spin particles in more than three spacetime dimensions. Our argument is insensitive to the physics above the effective cut-off scale and closes certain loopholes in previous arguments. Furthermore, it applies to higher spin particles even if they do not contribute to tree-level graviton scattering as a consequence of being charged under a global symmetry such as $\mathbb{Z}_2$. We derive analogous bounds in anti-de Sitter spacetime from analyticity properties of correlators of the dual CFT in the Regge limit. We also argue that an infinite tower of fine-tuned higher spin particles can still be consistent with causality. However, they necessarily affect the dynamics of gravitons at an energy scale comparable to the mass of the lightest higher spin particle. Finally, we apply the bound in de Sitter to impose restrictions on the structure of three-point functions in the squeezed limit of the scalar curvature perturbation produced during inflation.

[1]This chapter is based on N. Afkhami-Jeddi, S. Kundu and A. Tajdini, "A Bound on Massive Higher Spin Particles," arXiv:1811.01952 [hep-th]. I thank for several helpful discussions as well as comments. I also thank Nima Arkani-Hamed, Ibou Bah, Brando Bellazzini, James Bonifacio, Ted Jacobson, Marc Kamionkowski, David Kaplan, Jared Kaplan, Petr Kravchuk, David Meltzer, Joao Penedones, Eric Perlmutter, and David Simmons-Duffin for discussions.

4.1 Introduction

Weinberg in one of his seminal papers [112] showed that general properties of the S-matrix allow for the presence of the graviton. Not only that, the soft-theorem dictates that at low energies gravitons must interact universally with all particles – which is the manifestation of the equivalence principle in QFT. This remarkable fact has many far-reaching consequences for theories with higher spin particles.

Even in the early days of quantum field theory (QFT), it was known that there are restrictions on particles with spin $J > 2$ in flat spacetime. For example, Lorentz invariance of the S-matrix requires that massless particles interacting with gravity in flat spacetime cannot have spin more than two [112–114]. Moreover, folklore has it that any finite number of massive elementary higher spin particles, however fine-tuned, cannot interact with gravity in a consistent way. There is ample evidence suggestive of a strict bound on massive higher spin particles at least in flat spacetime in dimensions $D \geq 4$ from tree-level unitarity and asymptotic causality [5, 115–119],[2] however, to our knowledge there is no concrete argument which completely rules out a finite number of massive particles with spin $J > 2$.

Most notably, it was argued in [5] that in a theory with finite number of massive particles with spin $J > 2$, unless each higher spin particle is charged under a global symmetry such as $\mathbb{Z}_2$, they will contribute to eikonal scattering of particles, even with low spin ($J \leq 2$), in a way that violates asymptotic causality in flat spacetime. The same statement is true even in anti-de Sitter (AdS) spacetime where the global symmetries of higher spin particles are required by the chaos growth bound of the

[2]See comments in section 4.2.6 for comparison between arguments in the literature and the argument presented in this chapter.

dual CFT [41]. In addition, there is no known string compactification which leads to particles with spin $\ell > 2$ and masses $M \ll M_s$ in flat spacetime, where M_s is the string scale. Of course, it is well known that higher spin particles do exist in AdS, but they always come in an infinite tower and these theories become strongly interacting at low energies [120, 121]. All of these observations indicate that there are universal bounds on theories with higher spin massive particles. In this chapter, we will prove such a bound from causality. We will show that any finite number of massive elementary particles with spin $\ell > 2$, however fine tuned, cannot interact with gravitons in flat or AdS spacetimes (in $D \geq 4$ dimensions) in a way that is consistent with the QFT equivalence principle and preserves causality. In particular, we will demonstrate that the three-point interaction J-J-graviton must vanish for $J > 2$. However, this is one interaction that no particle can avoid due to the equivalence principle, implying that elementary particles with spin $J > 2$ cannot exist.

For massless higher spin particles, the inconsistencies are even more apparent. The tension between Lorentz invariance of the S-matrix and the existence of massless particles with spin $\ell > 2$ was already visible in [112]. Subsequently, the same tension was shown to exist for massless fermions with spin $\ell > 3/2$ [122,123]. A concrete manifestation of this tension is an elegant theorem due to Weinberg and Witten which states that any massless particle with spin $\ell > 1$ cannot possess a Lorentz covariant and gauge invariant energy-momentum tensor [113].[3] Of course, this theorem does not prohibit the existence of gravitons, rather it implies that the graviton must be fundamental. More recently, a generalization of the Weinberg-Witten theorem has been presented by Porrati which states that mass-

[3]See [124] for a nice review.

less particles with spin $\ell > 2$ cannot be minimally coupled to the graviton in flat spacetime [114]. Both of these theorems are completely consistent with various other observations made about interactions of massless higher spin particles in flat spacetime (see [125–130] and references therein). Furthermore, the generalized Weinberg-Witten theorem and the QFT equivalence principle are sufficient to completely rule out massless particles with spin $\ell > 2$ in flat spacetime [113, 114]. The basic argument is rather simple. The Weinberg-Witten theorem and its generalization by Porrati only allow non-minimal coupling between massless particles with spin $\ell > 2$ and the graviton. Whereas, it is well known that particles with low spin can couple minimally with the graviton. Therefore, the QFT equivalence principle requires that massless higher spin particles, if they exist, must couple minimally with the graviton at low energies – which directly contradicts the Weinberg-Witten/Porrati theorem.

Any well behaved Lorentzian QFT must also be unitary and causal.[4] Lorentz invariance alone was sufficient to rule out massless higher spin particles in flat spacetime. Whereas, massive elementary particles with spin $\ell > 2$ do not lead to any apparent contradiction with Lorentz invariance in flat spacetime. However, any such particle if present, must interact with gravitons. The argument presented in [5] implies that finite number of higher spin particles cannot be exchanged in any tree-level scattering. However, this restriction is not sufficient to rule out massive higher spin particles, rather it implies that each massive higher spin particle must be charged under $\mathbb{Z}_2$ or some other global symmetry. On the other hand, the equivalence principle requires the coupling between a single graviton and two spin-

[4]Unitarity and causality, as demonstrated in [1], are not completely unrelated in relativistic QFT.

ℓ particles to be non-vanishing. By considering an eikonal scattering experiment between scalars and elementary higher spin particles with spin ℓ and mass m in the regime $|s| \gg |t| \gg m$, where s and t are the Mandelstam variables, we will show that any such coupling between the higher spin particle and the graviton in flat spacetime leads to violation of asymptotic causality. This is accomplished by extending the argument of [5] to the scattering of higher spin particles which requires the phase shift to be non-negative for all choices of polarization of external particles.

A similar high energy scattering experiment can be designed in AdS to rule out elementary massive higher spin particles. However, we will take a holographic route which has several advantages. We consider a class of large-N CFTs in $D \geq 3$ dimensions with a sparse spectrum. The sparse spectrum condition, to be more precise, implies that the lightest single trace primary operator with spin $\ell > 2$ has dimension $\Delta_{\text{gap}} \gg 1$. It was first conjectured in [33] that this class of CFTs admit a universal holographic dual description with a low energy description in terms of Einstein gravity coupled to matter fields. The conjecture was based on the observation that there is a one-to-one correspondence between scalar effective field theories in AdS and perturbative solutions of CFT crossing equations in the $1/N$ expansion. The scalar version of this conjecture was further substantiated in [18–20,34–36,42,52,78–86,88] by using the conformal bootstrap. More recently, the conjecture has been completely proven at the linearized level even for spinning operators including the stress tensor [9, 25, 26, 62, 66, 131]. In the second half of the chapter, we will exploit this connection to constrain massive higher spin particles in AdS by studying large-N CFTs with a sparse spectrum. To this end, we introduced a new non-local operator, capturing the contributions to the Regge limit of the OPE of local operators. This operator is expressed as an integral of a local operator over a ball times a null-ray. It is obtained by generalizing the Regge OPE introduced in [131] to non-integer spins, resulting in an operator that is more naturally suited for parametrizing the contribution of Regge trajectories which require analytic continuation in both spin and scaling dimension.

In the holographic CFT side we will ask the dual question: is it possible to add an extra higher spin single trace primary operator with $\ell > 2$ and scaling dimension $\Delta \ll \Delta_{\text{gap}}$ and still get a consistent CFT? A version of this question has already been answered by a theorem in CFT that rules out any finite number of higher spin conserved currents [63,132–134]– which is the analog of the Weinberg-Witten theorem in AdS. However, ruling out massive higher spin particles in AdS requires a generalization of this theorem for non-conserved single trace primary operators of holographic CFTs. The chaos (growth) bound of Maldacena, Shenker, and Stanford [41] partially achieves this by not allowing any finite number of higher spin single trace primary operators to contribute as exchange operators in CFT four-point functions in the Regge limit. However, this restriction does not rule out the existence of such operators rather it prohibits these higher spin operators to appear in the operator product expansion (OPE) of certain operators. On the other hand, causality (chaos sign bound) imposes stronger constraints on non-conserved single trace primary operators. In particular, by using the holographic null energy condition (HNEC) [62,131] applied to correlators with external higher spin operators, we will show that massive higher spin fields in AdS (in $D \geq 4$ dimensions) lead to causality violation in the dual CFT. This implies that any finite number of massive elementary particles with spin $\ell > 2$ in AdS cannot be embedded in a well behaved UV theory in which the dynamics of gravitons at low energies is described by the Einstein-Hilbert action.

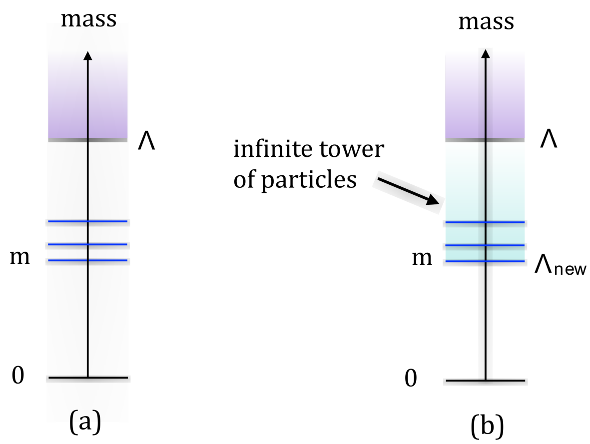


Figure 4.1: Spectrum of elementary particles with spin $J > 2$ in a theory where the dynamics of gravitons is described by the Einstein-Hilbert action at energy scales $E \ll \Lambda$. The cut-off scale Λ can be the string scale and hence there can be an infinite tower of higher spin particles above Λ. Figure (a) represents a scenario that also contains a finite number of higher spin particles below the cut-off and hence violates causality. Causality can only be restored if these particles are accompanied by an infinite tower of higher spin particles with comparable masses which is shown in figure (b). This necessarily brings down the cut-off scale to $\Lambda_{\text{new}} = m$, where m is the mass of the lightest higher spin particle.

One advantage of the holographic approach is that it also provides a possible solution to the causality problem. From the dual CFT side, we will argue that in a theory where the dynamics of gravitons is described by the Einstein-Hilbert action at energy scales $E \ll \Lambda$ (Λ can be the string scale M_s), a single elementary particle with spin $\ell > 2$ and mass $m \ll \Lambda$ violates causality unless the particle is accompanied by an infinite tower of (finely tuned) higher spin elementary particles with mass $\sim m$. Furthermore, causality also requires that these new higher spin particles (or at least an infinite subset of them) must be able to decay into two gravitons and hence modify the dynamics of gravitons at energy scales $E \sim m$. So, one can have a causal theory without altering the low energy behavior of gravity only if all the higher spin particles are heavier than the cut-off scale Λ.

Causality of CFT four-point functions in the lightcone limit also places non-trivial constraints on higher spin primary operators. In particular, it generalizes the Maldacena-Zhiboedov theorem of $D = 3$ [132] to higher dimensions by ruling out a finite number of higher spin conserved currents [63]. The advantage of the lightcone limit is that the constraints are valid for all CFTs – both holographic and non-holographic. However, the argument of [63] is not applicable when higher spin conserved currents do not contribute to generic CFT four-point functions as exchange operators. We will present an argument in the lightcone limit that closes this loophole by ruling out higher spin conserved currents even when none of the operators are charged under it.[5] For holographic CFTs, this completely rules out a finite number of massless higher spin particles in AdS in $D \geq 4$ dimensions.

The bound on higher spin particles has a natural application in inflation. If higher spin particles are present during inflation, they produce distinct signatures on the late time three-point function of the scalar curvature perturbation in the squeezed limit [135]. The bounds on higher spin particles in flat space and in AdS were obtained by studying local high energy scattering which is insensitive to the spacetime curvature. This strongly suggests that the same bound should hold even in de Sitter space.[6] Our bound, when applied in de Sitter, immediately implies that contributions of higher spins to the three-point function of the scalar

[5]We should note that we have not ruled out an unlikely scenario in which the OPE coefficients conspire in a non-trivial way to cancel the causality violating contributions. Three-point functions of conserved currents are heavily constrained by conformal invariance and hence this scenario is rather improbable.

[6]This argument parallels the argument made by Cordova, Maldacena, and Turiaci in [65]. The same point of view was also adopted in our previous paper [131].

curvature perturbation in the squeezed limit must be Boltzmann suppressed $\sim e^{-2\pi\Lambda/H}$, where H is the Hubble scale. Therefore, if the higher spin contributions are detected in future experiments, then the scale of new physics must be $\Lambda \sim H$. This necessarily requires the presence of not one but an infinite tower of higher spin particles with spins $\ell > 2$ and masses comparable to the Hubble scale. Any such detection can be interpreted as evidence in favor of string theory with the string scale comparable to the Hubble scale.

The rest of the chapter is organized as follows. In section 4.2, we present an S-matrix based argument to show that massive elementary particles with spin $J > 2$ cannot interact with gravitons in a way that preserves asymptotic causality. We derive the same bounds in AdS from analyticity properties of correlators of the dual CFT in section 4.3. In section 4.4, we argue that the only way one can restore causality is by adding an infinite tower of massive higher spin particles. In addition, we also discuss why stringy states in classical string theory are consistent with causality. Finally, in section 4.5, we apply our bound in de Sitter to constrain the squeezed limit three-point functions of scalar curvature perturbations produced during inflation.

4.2 Higher Spin Fields in Flat Spacetime

In this section, we explicitly show that interactions of higher spin particles with gravity lead to causality violation. Eikonal scattering has been used in the literature [5, 101, 102, 136–138] to impose constraints on interactions of particles with spin. When the center of mass energy is large and transfer momentum is small,

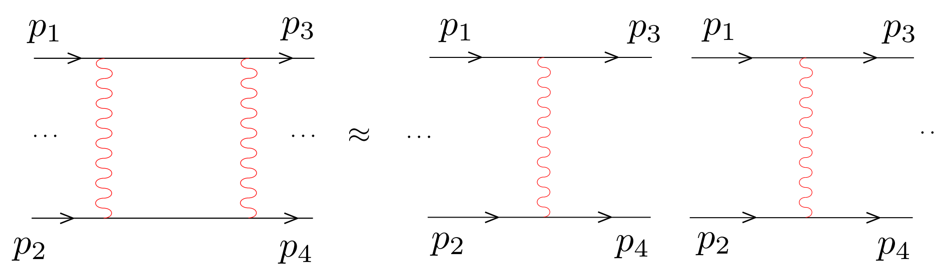


Figure 4.2: Tree-level exchange diagrams are the building blocks of ladder diagrams.

the scattering amplitude is captured by the eikonal approximation. Focusing on a specific exchange particle for now, the scattering amplitude is given by a sum of ladder diagrams. These diagrams can be resumed (see figure 4.2) and as a result introduce a phase shift in the scattering amplitude [139].[7] This phase shift produces a Shapiro time delay [140] that particles experience [5]. Asymptotic causality in flat spacetime requires the time delay and hence the phase shift to be non-negative [5, 44]. Moreover, positivity of the phase shift imposes restrictions on the tree-level exchange diagrams –which are the building blocks of ladder diagrams– constraining three-point couplings between particles. This method has been utilized to constrain three-point interactions between gravitons, massive spin-2 particles, and massless higher spin particles [5,101,102]. Here we apply the eikonal scattering method to external massive and massless elementary particles with spin $J > 2$.

We will briefly review eikonal scattering in order to explicitly relate the phase shift to the three-point interactions between elementary particles. We will take two of the external particles to be massive or massless higher spin particles ($J > 2$) and the other two particles to be scalars. The setup is shown in figure 4.3 where particles 1 and 3 are the higher spin particles, whereas particles 2 and 4 are scalars.

[7] We will comment more about the resummation later in the section.

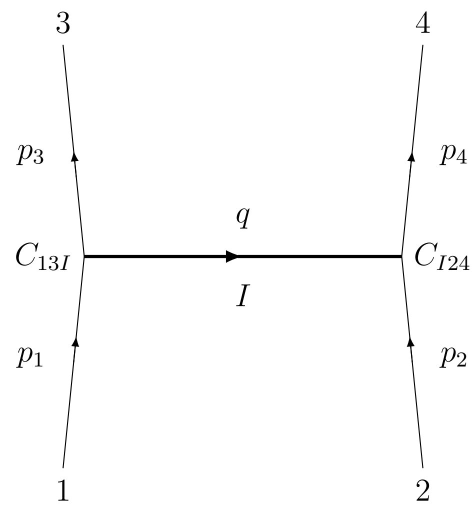


Figure 4.3: Eikonal scattering of particles. In this highly boosted kinematics, particles are moving almost in the null directions such that the center of mass energy is large.

We will then use on-shell methods to write down the general three-point interaction between higher spin elementary particles and gravitons [60]. This allows us to derive the most general form of the amplitude in the eikonal limit. Positivity of the phase shift for all choices of polarization tensors of external particles, constrains the coefficients of three-point vertices. In particular, for both massive and massless particles with spin $J > 2$ in spacetime dimensions $D \geq 4$, we find that the three-point interaction J-J-graviton must be zero. However, this is one interaction that no particle can avoid due to the equivalence principle, implying that elementary particles with spin $J > 2$ cannot exist.

4.2.1 Eikonal Scattering

Let us consider $2 \to 2$ scattering of particles in spacetime dimensions $D \geq 4$ as shown in figure 4.3. Coordinates are written in $\mathbb{R}^{1,D-1}$ with the metric

$$ds^2 = -dudv + d\vec{x}_\perp^2. \tag{4.2.1}$$

Denoting the momentum of particles by p_i, with i labeling particles 1 through 4, the Mandelstam variables are given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2 = -q^2, \tag{4.2.2}$$

where q is the momentum of the particle exchanged which in the eikonal limit has the property $q^2 = \vec{q}^2$, where $\vec{q}$ has components in directions transverse to the propagation of the external particles.[8] The tree level amplitude consists of the products of three-point functions[9]

$$M_{\text{tree}}(s, \vec{q}) = \sum_I \frac{C_{13I}(\vec{q}) C_{I24}(\vec{q})}{\vec{q}^{\,2} + m_I^2} , \tag{4.2.3}$$

where the sum is over all of the states of the exchanged particles with mass m_I. In the above expression, C_{13I} and C_{24I} are on-shell three-point amplitudes which are generally functions of the transferred momentum $\vec{q}$, as well as the polarization tensors and the center of mass variables.

In highly boosted kinematics, particles are moving almost in the null directions u and v with momenta P^u and P^v respectively. The center of mass energy s is large with respect to other dimensionful quantities such as the particle masses. In particular, we have $s \gg |t| = \vec{q}^{\,2}$. The total scattering amplitude is given by the

[8] See section 4.2.3 for the details of the kinematics.

[9] For a detail discussion about the $i\epsilon$ see [5].

sum of all ladder diagrams in t-channel which exponentiates when it is expressed in terms of the impact parameter $\vec{b}$ which has components only along the transverse plane,

$$iM_{\text{eik}}(s,-\vec{q}^{\,2}) = 2s\int d^{D-2}\vec{b}e^{-i\vec{q}\cdot\vec{b}}\left(e^{i\delta(s,\vec{b})}-1\right) \ , \tag{4.2.4}$$

where,

$$\delta(s,\vec{b}) = \frac{1}{2s}\int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}\, e^{i\vec{q}\cdot\vec{b}}M_{\text{tree}}(s,\vec{q}\,) \ . \tag{4.2.5}$$

Before we proceed, let us comment more on the exponentiation since it plays a central role in the positivity argument. We can interpret the phase shift as the Shapiro time-delay only when it exponentiates in the eikonal limit. However, it is known that the eikonal exponentiation fails for the exchange of particles with spin $J<2$ [141–143]. It is also not obvious that the tree level amplitude must exponentiate in the eikonal limit for the exchange of particles with spin $J\geq 2$. A physical argument was presented in [5] which suggests that for higher spin exchanges it is possible to get a final amplitude that is exponential of the tree level exchange diagram. First, let us think of particle 2 as the source of a shockwave and particle 1 to be a probe particle traveling in that background. At tree-level, the amplitude is given by $1+i\delta$, where we ensure that $\delta\ll 1$ by staying in a weakly coupled regime. Let us then send N such shockwaves so that we can treat them as individual shocks and hence the final amplitude, in the limit $\delta\to 0, N\to\infty$ with $N\delta$ =fixed, is approximately given by $(1+i\delta)^N\approx e^{iN\delta}$. This approximation is valid only if we can view N scattering processes as independent events. Moreover, we want to be in the weakly coupled regime. Both of these conditions can only be satisfied if δ grows with s – which is true for the exchange of particles with spin $J\geq 2$ [5]. Therefore, for higher-spin exchanges, we can interpret δ (or rather N times δ) as the Shapiro time delay of particle 1.

There is one more caveat. The exponentiation also depends on the assumption that δ is the same for each of the N-processes – in other words, the polarization of particle 3 is the complex conjugate of that of particle 1. In general, particle 3 can have any polarization, however, we can fix the polarization of particle 3 by replacing particle 1 by a coherent state of particles with a fixed polarization. Since we are in the weakly coupled regime, we can make the mean occupation number large without making δ large. This allows us to fix the polarization of particle 3 to be complex conjugate of that of particle 1 because of Bose enhancement (see [5] for a detail discussion).

Let us end this discussion by noting that the N-shock interpretation of the eikonal process is also consistent with classical gravity calculations. For example, the Shapiro time delay as obtained in GR from shockwave geometries is the same as the time delay obtained from the sum of all ladder diagrams for graviton exchanges – which indicates that these are the only important diagrams in the eikonal limit. Thus, it is reasonable to expect that the exponentiation of the tree-level diagram correctly captures the eikonal process.

Positivity:

When $\delta(s,\vec{b})$ grows with s, we can trust the eikonal exponentiation which allows us to relate the phase shift to time delay. In particular, for a particle moving in u direction with momentum $P^u > 0$, the phase shift $\delta(s,\vec{b})$ is related to the time delay of the particle by

$$\delta\left(s,\vec{b}\right) = P^u \Delta v \ . \tag{4.2.6}$$

Asymptotic causality in flat space requires that particles do not experience a time advance even when they are interacting [44]. Therefore, $\Delta v \geq 0$, implying that the phase shift must be non-negative as well.

So far our discussion is very general and it is applicable even when multiple exchanges contribute to the tree level scattering amplitude. From now on, let us restrict to the special case of massless exchanges.[10] Using the tree-level amplitude (4.2.3), we can write

$$
\begin{aligned}
\delta(s,\vec{b}) &= \frac{1}{2s}\sum_I \int \frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}\ e^{i\vec{q}\cdot\vec{b}}\frac{C_{13I}(\vec{q})C_{I24}(\vec{q})}{q^2} \\
&= \frac{\Gamma(\frac{D-4}{2})}{4\pi^{\frac{D-2}{2}}}\sum_I \frac{C_{I24}(-i\vec{\partial}_b)C_{13I}(-i\vec{\partial}_b)}{2s}\frac{1}{|\vec{b}|^{D-4}}
\end{aligned}
\tag{4.2.7}
$$

which must be non-negative. Note that $\vec{\partial}_b^2$ annihilates $1/|\vec{b}|^{D-4}$, which is why we can consider the exchange particle to be on-shell.[11]

4.2.2 Higher Spin-graviton Couplings

There are Lagrangian formulations of massive higher spin fields in flat spacetime, as well as in AdS [144–146]. However, in this section, we present a more general approach that does not require the knowledge of the Lagrangian. We write down all possible local three-point interactions between two higher spin elementary particles with spin J and a graviton. This three-point interaction is of importance for several reasons. First, this is one interaction that no particle can avoid because of

[10]For non-zero m_I, the $\vec{q}$ integral yields $(2\pi)^{\frac{2-D}{2}}(\frac{m_I}{b})^{\frac{D-4}{2}}K_{\frac{D-4}{2}}(m_I b)$, where K is the Bessel-K function.

[11]The same can be seen from the choice of the integration contour, as described in more detail in [5]. By rotating the contour of integration in $\vec{q}$, we cross the pole at $\vec{q}^{\,2}=0$ and hence it is sufficient to consider only three-point functions on-shell.

the equivalence principle. Therefore, the vanishing of this three-point interaction is sufficient to rule out existence of such higher spin particles. Moreover, as we will discuss later, this three-point interaction is sufficient to compute the full eikonal scattering amplitude between a scalar and a higher spin particle.

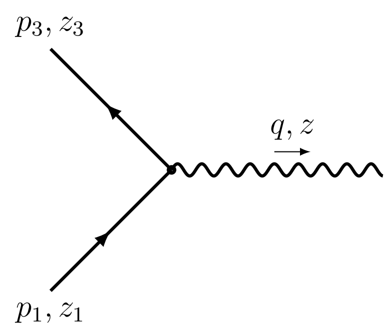


Figure 4.4: The three-point interaction between two elementary particles with spin J and a graviton.

We start with the massive case and consider the massless case later on. Here we use the same method used in [60, 101] for deriving the most general $J-J-2$ interaction. The momenta of higher spin particles are p_1, p_3 and the graviton has momentum q (see figure 4.4). The conservation and the on-shell conditions imply

$$p_1 = p_3 + q \ , \qquad p_1^2 = p_3^2 = -m^2 \ , \qquad q^2 = 0 \ , \tag{4.2.8}$$

where m is the mass of the higher spin particle. It is sufficient for us to consider polarization tensors which are made out of null and transverse polarization vectors z_1, z_3, z satisfying

$$z_1^2 = z_3^2 = z^2 = 0, \qquad z_1 \cdot p_1 = z_3 \cdot p_3 = z \cdot q = 0 \ . \tag{4.2.9}$$

Transverse symmetric polarization tensors can be constructed from null and transverse polarization vectors by substituting $z_i^{\mu_1} z_i^{\mu_2} \cdots z_i^{\mu_s} \to \mathcal{E}_i^{\mu_1\mu_2\cdots\mu_s}$ − traces. In addition, we need to impose gauge invariance for the graviton. This means that each on-shell vertex should be invariant under $z \to z + \alpha q$, where α is an arbitrary number. Using (4.2.8) and (4.2.9), we can write down all vertices in terms of only five independent building blocks[12]

$$\begin{aligned} &z_1 \cdot z_3 \ , \qquad z_1 \cdot q \ , \qquad z_3 \cdot q \ , \\ &z \cdot p_3 \ , \qquad (z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q) \ . \end{aligned} \tag{4.2.10}$$

In order to list all possible vertices for the interaction $J-J-2$, we must symmetrize the on-shell amplitudes under $1 \leftrightarrow 3$. We can then construct the most general form of on-shell three-point amplitude from these building blocks. In particular, for $J \geq 2$, we can write three distinct sets of vertices. The first set contains $J+1$ independent structures all of which are proportional to $(z \cdot p_3)^2$:

$$\begin{aligned} \mathcal{A}_1 &= (z \cdot p_3)^2 (z_1 \cdot z_3)^J \ , \\ \mathcal{A}_2 &= (z \cdot p_3)^2 (z_1 \cdot z_3)^{J-1} (z_3 \cdot q)(z_1 \cdot q) \ , \\ &\vdots \\ \mathcal{A}_{J+1} &= (z \cdot p_3)^2 (z_3 \cdot q)^J (z_1 \cdot q)^J \ . \end{aligned} \tag{4.2.11}$$

The second set contains J-independent structures which are proportional to $(z \cdot p_3)$:

$$\begin{aligned} \mathcal{A}_{J+2} &= (z \cdot p_3)((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_1 \cdot z_3)^{J-1}, \\ \mathcal{A}_{J+3} &= (z \cdot p_3)((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_1 \cdot z_3)^{J-2}(z_3 \cdot q)(z_1 \cdot q), \\ &\vdots \\ \mathcal{A}_{2J+1} &= (z \cdot p_3)((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_3 \cdot q)^{J-1}(z_1 \cdot q)^{J-1} \ . \end{aligned} \tag{4.2.12}$$

Finally the third set consists of $J-1$ independent structures which do not contain

[12] In $D = 4$, the collection of momentum and polarization vectors p_1, p_2, z_j $i, j = 1, 2, 3$ are not linearly independent and there are additional relations between the building blocks.

$(z \cdot p_3)$:

$$\begin{aligned}
\mathcal{A}_{2J+2} &= ((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))^2 (z_3 \cdot z_1)^{J-2}, \\
\mathcal{A}_{2J+3} &= ((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))^2 (z_3 \cdot z_1)^{J-3} (z_3 \cdot q)(z_1 \cdot q), \\
&\vdots \\
\mathcal{A}_{3J} &= ((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))^2 (z_3 \cdot q)^{J-2} (z_1 \cdot q)^{J-2} \ . \qquad (4.2.13)
\end{aligned}$$

In total there are $3J$ independent structures that contribute to the on-shell three-point amplitude of two higher spin particles with mass m and spin J and a single graviton. Therefore, the most general form of the three-point amplitude for $J \geq 1$, is given by[13]

$$C_{JJ2} = \sqrt{32\pi G_N} \sum_{n=1}^{3J} a_n \mathcal{A}_n. \qquad (4.2.14)$$

Note that $3J$ is also the number of independent structures in the three point functions in the CFT side after imposing permutation symmetry between operators $1, 3$ and taking conservation of stress-tensor into account.

4.2.3 Eikonal Kinematics

We now study the eikonal scattering of higher spin particles: $1, 2 \to 3, 4$, where, 1 and 3 label the massive higher spin particles with mass m and spin J and 2, 4 label scalars of mass m_s (see figure 4.3). Let us specify the details of the momentum and polarization tensors. In the eikonal limit, the momentum of particles are

[13] Here the propagators of the gravitons are canonically normalized to 1. Therefore, we need explicit G_N dependence in (4.2.14) since it couples to the graviton.

parametrized as follows[14]

$$
\begin{aligned}
p_1^\mu &= \left(P^u, \frac{1}{P^u}\left(\frac{\vec{q}^{\,2}}{4}+m_1^2\right), \frac{\vec{q}}{2}\right) \ , \qquad & p_3^\mu &= \left(\bar{P}^u, \frac{1}{\bar{P}^u}\left(\frac{\vec{q}^{\,2}}{4}+m_3^2\right), -\frac{\vec{q}}{2}\right) \ , \\
p_2^\mu &= \left(\frac{1}{P^v}\left(\frac{\vec{q}^{\,2}}{4}+m_2^2\right), P^v, -\frac{\vec{q}}{2}\right) \ , \qquad & p_4^\mu &= \left(\frac{1}{\bar{P}^v}\left(\frac{\vec{q}^{\,2}}{4}+m_4^2\right), \bar{P}^v, \frac{\vec{q}}{2}\right) \ ,
\end{aligned}
\tag{4.2.15}
$$

where, $P^u, \bar{P}^u, P^v, \bar{P}^v > 0$ and $p_1^\mu - p_3^\mu \equiv q$ is the transferred momentum of the exchange particle which is spacelike. The eikonal limit is defined as $P^u, P^v \gg |q|, m_i$. In this limit $P^u \approx \bar{P}^u, P^v \approx \bar{P}^v$ and the Mandelstam variable s is given by $s = -(p_1+p_2)^2 \approx P^u P^v$. Moreover, for our setup we have $m_1 = m_3 = m$ and $m_2 = m_4 = m_s$.

Massless particles have only transverse polarizations but massive higher spin particles can have both transverse and longitudinal polarizations. General polarization tensors can be constructed using the following polarization vectors

$$
\begin{aligned}
\epsilon_{T,\lambda}^\mu(p_1) &= \left(0, \frac{\vec{q}\cdot\vec{e}_\lambda^{\,(1)}}{P^u}, \vec{e}_\lambda^{\,(1)}\right) \ , \qquad & \epsilon_L^\mu(p_1) &= \left(\frac{P^u}{m}, \frac{1}{mP^u}\left(\frac{\vec{q}^{\,2}}{4}-m^2\right), \frac{\vec{q}}{2m}\right) \ , \\
\epsilon_{T,\lambda}^\mu(p_3) &= \left(0, -\frac{\vec{q}\cdot\vec{e}_\lambda^{\,(3)}}{P^u}, \vec{e}_\lambda^{\,(3)}\right) \ , \qquad & \epsilon_L^\mu(p_3) &= \left(\frac{P^u}{m}, \frac{1}{mP^u}\left(\frac{\vec{q}^{\,2}}{4}-m^2\right), -\frac{\vec{q}}{2m}\right) \ ,
\end{aligned}
\tag{4.2.16}
$$

where vectors $e_\lambda^\mu \equiv (0, 0, \vec{e}_\lambda)$ are complete orthonormal basis in the transverse direction $\vec{x}_\perp$. The longitudinal vectors do not satisfy (4.2.9) because $\epsilon_L \cdot \epsilon_L \neq 0$. However, they still form a basis for constructing symmetric traceless polarization tensors which are orthogonal to the corresponding momentum.

The polarization tensors constructed from (4.2.16) are further distinguished by their spin under an $SO(D-2)$ rotation group which preserves the longitudinal

[14]Our convention is $p^\mu = (p^u, p^v, \vec{p})$.

polarization ϵ_L for each particle. We denote this basis of polarization tensors as $\mathcal{E}_j^{\mu_1\mu_2\cdots\mu_J}(p_i)$ where j labels the spin under $SO(D-2)$. These tensors are basically organized by the number of transverse polarization vectors they contain. The most general polarization tensor for a particle with spin J can now be decomposed as

$$\boldsymbol{\mathcal{E}}^{\mu_1\cdots\mu_J}(p) = \sum_{j=0}^{J} r_j \mathcal{E}_j^{\mu_1\cdots\mu_J}(p), \tag{4.2.17}$$

where r_j's are arbitrary complex numbers. However, in order to show that the higher spin particles cannot interact with gravity in a consistent way, we need only to consider a subspace spanned by

$$\mathcal{E}_J^{\mu_1\mu_2\cdots\mu_J} = \epsilon_{T,\lambda_1}^{\mu_1}\epsilon_{T,\lambda_2}^{\mu_2}\cdots\epsilon_{T,\lambda_J}^{\mu_J}, \tag{4.2.18}$$

$$\mathcal{E}_{J-1}^{\mu_1\mu_2\cdots\mu_J} = \sqrt{J}\epsilon_L^{(\mu_1}\epsilon_{T,\lambda_2}^{\mu_2}\epsilon_{T,\lambda_3}^{\mu_3}\cdots\epsilon_{T,\lambda_J}^{\mu_J)}, \tag{4.2.19}$$

$$\mathcal{E}_{J-2}^{\mu_1\mu_2\cdots\mu_J} = \sqrt{\frac{D-1}{D-2}}\left(\epsilon_L^{(\mu_1}\epsilon_L^{\mu_2} - \frac{\mathcal{P}^{\mu_1\mu_2}}{D+2J-5}\right)\epsilon_{T,\lambda_3}^{\mu_3}\epsilon_{T,\lambda_4}^{\mu_4}\cdots\epsilon_{T,\lambda_J}^{\mu_J)}, \quad \mathcal{P}^{\mu\nu} \equiv \eta^{\mu\nu} + \frac{p^\mu p^\nu}{m^2}, \tag{4.2.20}$$

where, after contractions with other tensors we perform the following substitution: $e_{\lambda_1}^{i_1}e_{\lambda_2}^{i_2}\cdots e_{\lambda_j}^{i_j} \to e^{i_1\cdots i_j}$ in which $e^{i_1\cdots i_j}$ is a transverse symmetric traceless tensor.[15] One can easily continue this construction to generate the remaining polarization tensors. One should add more longitudinal polarization vectors and subtract traces in order to make them traceless.

[15] In other words, whenever we see a combination of transverse polarization vectors: $\epsilon_{T,\lambda_1}^{\mu_1}\epsilon_{T,\lambda_2}^{\mu_2}\cdots\epsilon_{T,\lambda_S}^{\mu_S}$, we will replace that by either of $\epsilon_{T,+}^{\mu_1}\epsilon_{T,+}^{\mu_2}\cdots\epsilon_{T,+}^{\mu_S} \pm \epsilon_{T,-}^{\mu_1}\epsilon_{T,-}^{\mu_2}\cdots\epsilon_{T,-}^{\mu_S}$, where $e_+^\mu \equiv (0,0,1,i,\vec{0})$ and $e_-^\mu \equiv (0,0,1,-i,\vec{0})$. For us, it is sufficient to restrict to these set of polarization tensors.

4.2.4 Bounds on Coefficients

We now have all the tools we need to utilize the positivity condition (4.2.7) in the eikonal scattering of a massive higher spin particle and a scalar. The expression (4.2.7) requires knowledge of the contributions of all the particles that can be exchanged. However, as we explain next, in the eikonal limit the leading contribution is always due to the graviton exchange. Let us explain this by discussing all possible exchanges:

- Graviton exchange: Since, gravitons couple to all particles, the scattering amplitude in the eikonal limit will always receive contributions from graviton exchanges. In particular, in the eikonal limit, the contribution of graviton exchange to the phase shift goes as $\delta(s,b) \sim s$.
- Exchange of particles with spin $J < 2$: These exchanges are always subleading in the eikonal limit and hence can be ignored.[16]
- Exchange of higher spin particles $J > 2$: In the eikonal limit, the exchange of a particle with spin J can produce a phase shift $\delta(s,b) \sim s^{J-1}$. However, it was shown in [5] that a phase shift that grows faster than s leads to additional causality violation. Therefore, if higher spin particles are present, their interactions must be tuned in such a way that they cannot be exchanged in eikonal scattering. This happens naturally when each higher spin particle is individually charged under a global symmetry such as $\mathbb{Z}_2$. We should note that it is possible to have a scenario in which an infinite tower of higher spin particles can be exchanged without violating causality. However, we will

[16] We have mentioned before that the eikonal exponentiation fails for the exchange of particles with spin $J < 2$. However, we can still ignore them because the exchange of lower spin particles cannot compete with the graviton exchange in the eikonal limit.

restrict to the case where only a finite number of higher spin particles are present. At this point, let us also note that in AdS, the exchange of a finite number of higher spin particles are ruled out by the chaos growth bound of the dual CFT.

- Exchange of massive spin-2 particles: Massive spin-2 particles can be present in nature. However, the exchange of these particles, as explained in [5], cannot fix the causality violation caused by the graviton exchange. Therefore, without any loss of generality, we can assume that the scalar particles do not interact with any massive spin-2 particle. For now this will allow us to ignore massive spin-2 exchanges. Let us note that it is not obvious that the argument of [5] about massive spin-2 exchanges necessarily holds for scattering of higher spin particles. So, at the end of this section, we will present an interference based argument to explain the reason for why even an infinite tower of massive spin-2 exchanges cannot restore causality.

In summary, in the eikonal limit, it is sufficient to consider only the graviton exchange. In fact, we can safely assume that the scalar interacts with everything, even with itself, only via gravity. Let us also note that we are studying eikonal scattering of higher spin particles with scalars only for simplicity. The calculations as well as the rest of the arguments are almost identical even if we replace the scalar by a graviton. In the graviton case, the argument of [5] about massive spin-2 exchanges holds – this implies that the presence of massive spin-2 particles will not change our final bounds.

We now use (4.2.7) to calculate the phase shift where C_{13I} is given by equation

(4.2.14). For scalar-scalar-graviton there is only one vertex, written as

$$C_{I24} \equiv C_{002} = \sqrt{32\pi G}(z \cdot p_2)^2 \ . \tag{4.2.21}$$

Consequently, the sum in (4.2.7) is over the polarization of the exchanged graviton. In the eikonal limit, this sum receives a large contribution from only one specific intermediate state corresponding to the polarization tensor of the exchanged graviton appearing in C_{13I} of the form $z^v z^v$ and the polarization tensor appearing in C_{I24} of the form $z^u z^u$.[17]

As discussed earlier, if $\delta(s, \vec{b})$ grows with s, causality requires $\delta(s, \vec{b}) \geq 0$ as a condition which must be true independent of polarization tensors we choose for our external particles. In particular, in the basis $\boldsymbol{\mathcal{E}}$, $\delta(s, \vec{b})$ can be written as

$$\delta(s, \vec{b}) = \boldsymbol{\mathcal{E}}_1^\dagger \mathcal{K}(\vec{b}) \boldsymbol{\mathcal{E}}_3, \tag{4.2.23}$$

where $\mathcal{K}$ is a Hermitian matrix which is encoding the eikonal amplitude in terms of the structures written in (4.2.14).[18] Causality then requires $\mathcal{K}$ to be a positive semi-definite matrix for any $\vec{b}$. We sketch the argument for constraining three-point interactions here and leave the details to appendices A.10 and A.11.

First, let us discuss $D > 4$.[19] We start with the general expressions for on-shell three point amplitudes. The polarization tensors for both particles 1 and 3 are chosen to be in the subspace spanned by $\mathcal{E}_J, \mathcal{E}_{J-1}$ and $\mathcal{E}_{J-2}$:

$$\boldsymbol{\mathcal{E}} = r_J \mathcal{E}_J + r_{J-1} \mathcal{E}_{J-1} + r_{J-2} \mathcal{E}_{J-2} \ , \tag{4.2.24}$$

[17] In the eikonal limit, the sum over the polarization of the graviton, in general, is given by [5]

$$\sum_I \epsilon^I_{\mu\nu}(q)(\epsilon^I_{\rho\sigma}(q))^* \sim \frac{1}{2} \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\nu\rho}\eta_{\mu\sigma}\right) \ . \tag{4.2.22}$$

[18] This assumes polarization tensors being properly normalized, i.e. $\boldsymbol{\mathcal{E}}_i^\dagger \boldsymbol{\mathcal{E}}_i = 1$, otherwise (4.2.23) should be divided by $\boldsymbol{\mathcal{E}}_1^\dagger \boldsymbol{\mathcal{E}}_3$.

[19] $D = 4$ is more subtle for various reasons and we will discuss it separately.

where, r_J, r_{J-1} and r_{J-2} are real numbers. Using eikonal scattering we organize the phase shift in the small b limit in terms of the highest negative powers of the impact parameter b. We start by setting $r_{J-2} = 0$. We then demand $\mathcal{K}(\vec{b})$ to have non-negative eigenvalues order by order in $1/b$ for transverse polarization $e^{\oplus}$ (or $e^{\otimes}$) for all directions of the impact parameter $\vec{b}$.[20] This imposes the following constraints on the coefficients

$$a_i = 0 \ , \qquad i \in \{2, 3, \cdots 3J\} \setminus \{J+2, 2J+2\} \ , \tag{4.2.25}$$

where, a_i is defined in (4.2.14). In other words, we find that all vertices with more than two derivatives must vanish. Moreover, the coefficients a_1, a_{J+2}, a_{2J+2} are related and the interaction C_{JJ2} can be reduced to the following vertex

$$\begin{aligned} C_{JJ2} =& a_1 (z_1 \cdot z_3)^{J-2} \Big((z_1 \cdot z_3)^2 (z \cdot p_3)^2 + J \big((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q) \big) (z_1 \cdot z_3)(z \cdot p_3) \\ &+ \frac{J(J-1)}{2} \big((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q) \big)^2 \Big) \ . \end{aligned} \tag{4.2.26}$$

When $J = 2$, no further constraints can be obtained using any other choice of polarization tensors. On the other hand, for $J > 2$ we can use the polarization tensor $\mathcal{E}_{J-2}$ (which always exists for $J \geq 2$) yielding

$$a_1 = 0 \ , \tag{4.2.27}$$

implying that $C_{JJ2} = 0$. Therefore, there is no consistent way of coupling higher spin elementary particles with gravity in flat spacetime in $D > 4$ dimensions.[21]

[20] Transverse polarizations $e^{\otimes}, e^{\oplus}$ are given explicitly in appendix A.10.

[21] There are parity odd structures in $D = 5$ for massive particles of any spin. As we show in appendix A.12, These interactions also violate causality for $J > 2$ as well as $J \leq 2$.

4.2.5 $D = 4$

The $D = 4$ case is special for several reasons. First of all, the $3J$ structures of on-shell three-point amplitude of two higher spin particles with mass m and spin J and a single graviton are not independent in $D = 4$. These structures are built out of 5 vectors, however, in $D = 4$, any 5 vectors are necessarily linearly dependent. In particular, one can show that

$$m^2B^2 + 2AB(q\cdot z_3)(q\cdot z_1) + 2A^2(q\cdot z_3)(q\cdot z_1)(z_1\cdot z_3) = 0\ , \tag{4.2.28}$$

where, $A = (z\cdot p_3)$ and $B = (z\cdot z_3)(z_1\cdot q) - (z\cdot z_1)(z_3\cdot q)$ are two of the building blocks of on-shell three-point amplitudes. The above relation implies that structures in the set (4.2.13) in $D = 4$ are not independent since they can be written as structures from set (4.2.11) and (4.2.12). Therefore, for spin J in $D = 4$, there are $2J + 1$ independent structures which is in agreement with the number of independent structures in the CFT three point function of the stress tensor and two spin-J non-conserved primary operators. The $D = 4$ case is special for one more reason – there are parity odd structures for any spin J. In order to list all possible parity odd vertices for the interaction $J - J - 2$, we introduce the following building block that does not preserve parity :

$$\mathcal{B} = \epsilon^{\mu_1\mu_2\mu_3\mu_4} z_{1_{\mu_1}} z_{3_{\mu_2}} z_{\mu_3} q_{\mu_4}\ . \tag{4.2.29}$$

The parity odd on-shell three-point amplitude can be constructed using this building block. In particular, we can write two distinct sets of vertices with $\mathcal{B}$. The first set contains J independent structures:

$$
\begin{aligned}
&\mathcal{A}_1^{odd} = \mathcal{B}(z \cdot p_3)(z_1 \cdot z_3)^{J-1} \ , \\
&\mathcal{A}_2^{odd} = \mathcal{B}(z \cdot p_3)(z_1 \cdot z_3)^{J-2}(z_3 \cdot q)(z_1 \cdot q) \ , \\
&\vdots \\
&\mathcal{A}_J^{odd} = \mathcal{B}(z \cdot p_3)(z_3 \cdot q)^{J-1}(z_1 \cdot q)^{J-1} \ .
\end{aligned} \tag{4.2.30}
$$

The second set contains $J-1$ independent structures:

$$
\begin{aligned}
&\mathcal{A}_{J+1}^{odd} = \mathcal{B}((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_1 \cdot z_3)^{J-2}, \\
&\mathcal{A}_{J+2}^{odd} = \mathcal{B}((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_1 \cdot z_3)^{J-3}(z_3 \cdot q)(z_1 \cdot q), \\
&\vdots \\
&\mathcal{A}_{2J-1}^{odd} = \mathcal{B}((z \cdot z_3)(z_1 \cdot q) - (z \cdot z_1)(z_3 \cdot q))(z_3 \cdot q)^{J-2}(z_1 \cdot q)^{J-2} \ .
\end{aligned} \tag{4.2.31}
$$

In $D=4$, there is another parity odd structure which is not related to the above structures and hence should be considered independent[22]

$$
\mathcal{A}_{2J}^{odd} = \epsilon^{\mu_1\mu_2\mu_3\mu_4} z_{1\mu_1} p_{1\mu_3} z_{3\mu_3} p_{3\mu_4} (z \cdot p_3)^2 (z_3 \cdot q)^{J-1}(z_1 \cdot q)^{J-1} \ . \tag{4.2.32}
$$

Therefore, the most general form of the three-point amplitude for $J \geq 1$ is given by

$$
C_{JJ2} = \sqrt{32\pi G_N} \left(\sum_{n=1}^{2J+1} a_n \mathcal{A}_n + \sum_{n=1}^{2J} \bar{a}_n \mathcal{A}_n^{odd} \right) \ . \tag{4.2.33}
$$

We can again use the polarization tensors (4.2.18) to derive constraints. However, for $D=4$ the setup of this section is not adequate to completely rule out particles with $J>2$. In $D=4$, the transverse space is only two-dimensional and

[22] We would like to thank J. Bonifacio for pointing this out.

therefore does not provide enough freedom to derive optimal bounds. In particular, we find that a specific non-minimal coupling is consistent with the positivity of the phase shift. We eliminate this remaining non-minimal coupling by considering interference between the graviton and the higher spin particle.

In $D = 4$, the use of the polarization tensors (4.2.18) leads to the following bounds: $\bar{a}_n = 0$ and $a_2, \cdots, a_{2J+1}$ are fixed by a_1 (see (A.11.15)). The same set of bounds can also be obtained by using a simple null polarization vector

$$\epsilon^\mu(p_1) = i\epsilon^\mu_L(p_1) + \epsilon^\mu_{T,\hat{x}}(p_1) \ , \qquad \epsilon^\mu(p_3) = -i\epsilon^\mu_L(p_3) + \epsilon^\mu_{T,\hat{x}}(p_3) \ , \tag{4.2.34}$$

where the transverse and longitudinal vectors are defined in (4.2.16) and the vector $\hat{x}$ is given by $\hat{x} = (0, 0, 1, 0)$. The phase-shift in $D = 4$ is

$$\delta(s, \vec{b}) = \frac{1}{4\pi s} \sum_I C_{I24}(-i\vec{\partial}_b) C_{13I}(-i\vec{\partial}_b) \ln\left(\frac{L}{b}\right) \ , \tag{4.2.35}$$

where, L is the IR regulator. Introduction of the IR regulator is necessary because of the presence of IR divergences in $D = 4$. Using the polarization (4.2.34) we obtain

$$\delta(s, \vec{b}) \sim s a_1 \ln\left(\frac{L}{b}\right) + s \sum_{n=0}^{2J-1} \frac{1}{b^{2J-n}} \left(f_n \cos((2J-n)\theta) + \bar{f}_n \sin((2J-n)\theta) \right) \ , \tag{4.2.36}$$

where, $\cos\theta = \hat{b} \cdot \hat{x}$. Coefficients f_n and $\bar{f}_n$ are linear combinations of parity even and parity odd coupling constants respectively. Requiring the phase shift to be positive order by order in $1/b$ in the limit $b \ll 1/m$ imposes the condition $f_n = \bar{f}_n = 0$. This implies that all the parity odd couplings must vanish and all the parity even couplings are completely fixed once we specify a_1 (full set of constraints for spin J are shown in (A.11.15).) Therefore, positivity of the phase shift (4.2.36) is consistent with a specific non-minimal coupling of higher spin particles in $D = 4$.

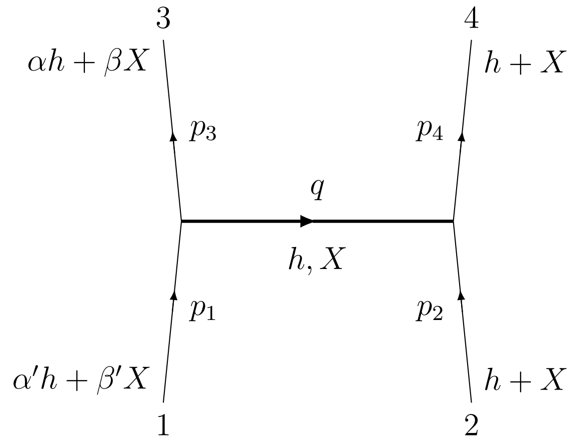


Figure 4.5: Bounds from interference in $D = 4$. In-states are linear combinations of massive higher spin particle X and the graviton h.

In order to rule out this specific interaction, we now consider interference between the graviton and the higher spin particle.

Bound from Interference

We now consider eikonal scattering of gravitons and massive higher spin particles: $1, 2 \to 3, 4$. In this setup, 1 and 3 are linear combinations of massive higher spin particle X and the graviton: $\alpha h + \beta X$ and $\alpha' h + \beta' X$ respectively, where $\alpha, \alpha', \beta, \beta'$ are arbitrary real coefficients. While 2 and 4 are a fixed combination of X and the graviton: $h + X$. We will treat 2 as the source and 1 as the probe (see figure 4.5). This setup is very similar to the setup of [102].

Positivity of the phase-shift can now be expressed as semi-definiteness of the following matrix

$$\begin{pmatrix} \delta_{hh} & \delta_{hX} \\ \delta_{Xh} & \delta_{XX} \end{pmatrix} \succeq 0 \ , \tag{4.2.37}$$

where, δ_{Xh} represents phase-shift when particle 1 is a higher spin particle of mass m and spin J and particle 3 is a graviton.[23] The above condition can also be restated as an interference bound

$$|\delta_{Xh}|^2 \leq \delta_{hh}\delta_{XX} \ , \tag{4.2.38}$$

where we have used the fact that $\delta_{Xh} = \delta^*_{hX}$. In the eikonal limit, the dominant contribution to both δ_{hh} and δ_{XX} comes from the graviton exchange and hence $\delta_{hh}, \delta_{XX} \sim s$, where s is the Mandelstam variable. Therefore, asymptotic causality requires that δ_{Xh} should not grow faster than s.

Let us now compute δ_{Xh} for a specific configuration. Momenta of the particles are again given by (4.2.15) with appropriate masses. Moreover, we will use the following null polarization vectors for various particles:

$$\begin{aligned} &\epsilon^\mu_X(p_1) = i\epsilon^\mu_L(p_1) + \epsilon^\mu_{T,\hat{x}}(p_1) \ , \qquad &&\epsilon^\mu_h(p_3) = \epsilon^\mu_{T,\hat{x}}(p_3) + i\epsilon^\mu_{T,\hat{y}}(p_3) \ , \\ &\epsilon^\mu_X(p_2) = i\epsilon^\mu_L(p_2) + \epsilon^\mu_{T,\hat{x}}(p_2) \ , \qquad &&\epsilon^\mu_h(p_2) = \epsilon^\mu_{T,\hat{x}}(p_2) - i\epsilon^\mu_{T,\hat{y}}(p_2) \ , \\ &\epsilon^\mu_X(p_4) = -i\epsilon^\mu_L(p_4) + \epsilon^\mu_{T,\hat{x}}(p_4) \ , \qquad &&\epsilon^\mu_h(p_4) = \epsilon^\mu_{T,\hat{x}}(p_4) + i\epsilon^\mu_{T,\hat{y}}(p_4) \ , \end{aligned} \tag{4.2.39}$$

where $\hat{x} = (0,0,1,0)$ and $\hat{y} = (0,0,0,1)$. In the eikonal limit the dominant contribution to δ_{Xh} comes from X-exchange. In particular, after imposing constraints (A.11.15), we find that

$$\delta_{Xh} \sim a_1 s^{J-1} \frac{e^{-2i(J-2)\theta}}{b^{2(J-2)} m^{4(J-2)}} \ , \tag{4.2.40}$$

[23]similar notation is used for other elements of the phase-shift matrix.

where $\cos\theta = \hat{b}\cdot\hat{x}$. The above phase-shift violates causality for $J > 2$ implying

$$a_1 = 0 \qquad \text{for} \qquad J > 2 \; . \tag{4.2.41}$$

Therefore, there is no consistent way of coupling higher spin elementary particles with gravity even in four dimensional flat spacetime.

4.2.6 Comments

Comparison with other arguments

As mentioned in the introduction, there are qualitative arguments in the literature in $D = 4$ suggesting that elementary massive higher spin particles cannot exist. The idea originally advocated by Weinberg, is to require physical theories for elementary particles to have a well behaved high energy limit or equivalently to demand a smooth limit for the amplitude as $m_X \to 0$ [116, 117]. However, for minimal coupling with spin $J > 2$ particles, the amplitude grows with powers of $\left(\frac{s}{m_X^2}\right)$ as $m_X \to 0$ [115]. Therefore, given a fixed and finite cutoff scale Λ and a mass m_X, the amplitude can become $\mathcal{O}(1)$ for $m_X \ll \sqrt{s} \ll \Lambda$. For instance, it was shown in [118] by considering only the minimal coupling of spin $\frac{5}{2}$ to gravity, that tree-level unitarity breaks down at the energy $\sqrt{s} \sim \sqrt{m_X M_{pl}} \ll M_{pl}$. Moreover, the break-down scale for a particle of spin J was conjectured to be even lower $\sim \left(m_X^{2J-2} M_{pl}\right)^{\frac{1}{2J-1}}$ [147]. This was shown to be true for massive spin $J = 2$ particles [148]. The existence of this scale implies that this particle cannot exist if tree-level unitarity is required to persist for scales up to M_{pl}. This seems natural if we require the theory of higher spin fields to be renormalizable. However, from an effective field theory point of view, the smooth $m_X \to 0$ requirement, determines only the range of masses and cut-off scales over which the low energy tree level amplitude is a good description of this massive higher spin scattering experiment. Note that even within the tree level unitarity arguments, one still needs to consider all possible non-minimal couplings as well as all contact interactions in order to ensure that they do not conspire to change the singular behavior of the amplitude in the $m_X \to 0$ limit. In fact, [118,119] demonstrates examples in which adding non-minimal couplings can change the high energy singular behavior of the amplitude for longitudinal part of polarizations.

By contrast, the causality arguments used here, require only the cut-off to be parametrically larger than the mass of the higher spin particle, $\Lambda \gg m_X$. Then, given an impact parameter $b \ll m_X^{-1}$, the desired bounds are obtained even if the amplitude or phase shift $M(s,t)$, $\delta(s,b) \ll 1$ (unlike the violation of tree-level unitarity requiring the amplitude to be $\mathcal{O}(1)$) since even the slightest time advance is forbidden by causality. Moreover, in the eikonal experiment, the two incoming particles do not overlap and hence contributions from the other channel and contact diagrams can be ignored [5].

An Interference Argument for $D > 4$

A generalization of the interference argument of $D = 4$ to higher dimensions also suggests that there is tension between massive higher spin particles and asymptotic causality. In fact, it might be possible to derive the bounds of this section by demanding that the phase shift δ_{Xh} does not grow faster than s, however, we have not checked this explicitly. This argument has one immediate advantage. For a particle with spin J, $\delta_{Xh} \sim s^{J-1}$ and therefore it is obvious that even an infinite tower of massive spin-2 exchanges cannot restore causality. The only way causality can be restored is if we add an infinite tower of massive higher spin particles. We should note that this arguments rely on the additional assumption that the eikonal approximation is valid for spin-J exchange with $J > 2$. The N-shocks argument of [5] is also applicable here which strongly suggests that the eikonal exponentiation holds even for $J > 2$, however, a rigorous proof is still absent.

Massless Case

Higher spin massless particles are already ruled out by the Weinberg-Witten theorem. Nonetheless, we can rederive this fact using the eikonal scattering setup. If the higher spin particles are massless, then gauge invariance requires that each vertex is invariant under the shift $z_i \to z_i + \alpha_i\, p_i$, where α_i's are arbitrary real numbers. In this case only the three following structures are allowed for $J \geq 2$

$$\begin{aligned}
\mathcal{D}_1 &= (z \cdot p_3)^2 (z_1 \cdot q)^J (z_3 \cdot q)^J, \qquad (4.2.42)\\
\mathcal{D}_2 &= ((z_3 \cdot q)(z_1 \cdot z) - (z \cdot z_3)(z_1 \cdot q) - (z \cdot p_3)(z_1 \cdot z_3))(z \cdot p_3)(z_1 \cdot q)^{J-1}(z_3 \cdot q)^{J-1},\\
\mathcal{D}_3 &= ((z_3 \cdot q)(z_1 \cdot z) - (z \cdot z_3)(z_1 \cdot q) - (z \cdot p_3)(z_1 \cdot z_3))^2 (z_1 \cdot q)^{J-2}(z_3 \cdot q)^{J-2}\ .
\end{aligned}$$

This is again, as we will see in the next section, in agreement with the three structures appearing in the CFT three point function once we impose conservation constraints for all three operators. The general form of the three-point function for $J \geq 2$ is now given by

$$C_{JJ2} = \sqrt{32\pi G_N} \sum_{n=1}^{3} d_n \mathcal{D}_n\ . \qquad (4.2.43)$$

For massless particles, $\mathcal{E}_J$ is the only polarization tensor. As before, by requiring asymptotic causality we find

$$d_n = 0 \qquad n = 1, 2, 3 \tag{4.2.44}$$

for $J > 2$.

Parity Violating Interactions of Massive Spin-2 in $D = 4$

The argument presented in this section can also be applied to $J = 2$ in $D \geq 4$. Of course, our argument does not rule out massive spin-2 particles. Rather it restricts the coupling between two massive spin-2 particles and a graviton to be minimal (4.2.26) which agrees with [102]. However, for $D = 4$ our argument does rule out parity violating interactions between massive spin-2 particles and the graviton. Moreover, the same conclusion about parity violating interactions holds even for massive spin-1.

Restoration of Causality

Let us now discuss the possible ways of bypassing the arguments presented in this section. Our arguments utilized the eikonal limit $m, q \ll \sqrt{s} \ll \Lambda$, where Λ is the UV cut-off of the theory. Hence, our argument breaks down if the mass of the higher spin particle $m \sim \Lambda$.

There is another interesting possibility. One can have a massive higher spin particle with mass $m \ll \Lambda$ and causality is restored by adding one or more additional particles. The contribution to the phase shift for a tree level exchange of a particle of mass $M \gg m$, $\frac{1}{b}$ is exponentially suppressed $\sim e^{-bM}$. Hence, these additional contributions can be significant enough if the masses of these particles are not much larger than m. In addition, exchange of these additional particles can only restore causality if they have spin $J > 2$. However, exchange of any finite number of such particles will lead to additional causality violation. Hence, the only possible way causality can be restored is by adding an infinite tower of fine-tuned higher spin particles with masses comparable to m. Furthermore, causality for the scattering J+graviton$\rightarrow$ J+graviton also requires that an infinite subset of these new higher spin particles must be able to decay into two gravitons which implies that this infinite tower does affect the dynamics of gravitons at energies $\sim m$.[24] We will discuss this in more detail in section 4.4.

Composite Higher Spin Particles

The argument of this section is applicable to elementary massive higher spin particles. However, whether a particle is elementary or not must be understood from the perspective of effective field theory. Hence, the argument of this section is also applicable to composite higher spin particles as long as they look elementary enough at a certain energy scale. In particular, if the mass of a composite particle is m but it effectively behaves like an elementary particle up to some energy scale Λ which is parametrically higher than m, then the argument of this section is still applicable. More generally, argument of this section rules out any composite higher spin particle which is isolated enough such that it does not decay to other particles after interacting with high energy gravitons $q \gg m$.

[24]Note that we ignored loops of the higher spin tower. From the scattering J+graviton$\rightarrow$ J+graviton, it is clear that an infinite tower of higher spin particles with mass $M \gg m$ cannot restore causality even if we consider loops.

Validity of the Causality Condition

Let us end this section by mentioning a possible caveat of our argument. In this section, we have shown that presence of massive higher spin particles is inconsistent with asymptotic causality which requires that particles do not experience a time advance even when they interact with each other. It is believed that any Lorentzian QFT must obey this requirement. However, there is no rigorous S-matrix based argument that shows that positivity of the time delay is a necessary requirement of any UV complete theory. A physical argument was presented in [5] which relates positivity of the phase shift to unitarity but it would be nice to have a more direct derivation. In the next section, we present a CFT-based derivation of the same bounds in anti-de Sitter spacetime which allows us to circumvent this technical loophole.

4.3 Higher Spin Fields in AdS$_D$

Let us now consider large-N CFTs in dimensions $D \geq 3$ with a sparse spectrum. CFTs in this class are special because at low energies they exhibit universal, gravity-like behavior. This duality allows us to pose a question in the CFT in D-dimensions which is dual to the question about higher spin fields in AdS in $D = d + 1$ dimensions. Is it possible to have additional higher spin single trace primary operators X_ℓ with $\ell > 2$ and scaling dimension $\Delta \ll \Delta_{\text{gap}}$ in a holographic CFT?

In general, any such operator X_ℓ will appear as an exchange operator in a four-point function of even low spin operators. In the Regge limit $\sigma \to 0$,[25] the contribution to the four-point function from the X_ℓ-exchange goes as $\sim 1/\sigma^{\ell-1}$ which violates the chaos growth bound of [41] for $\ell > 2$ and hence all CFT three-point functions $\langle X_\ell OO \rangle$ must vanish for any low spin operator O. In the gravity side, this rules out all bulk couplings of the form $\mathcal{O}\mathcal{O}\mathcal{X}_\ell$ in AdS, where $\mathcal{X}_\ell$ is a higher spin bulk field (massive or massless) and $\mathcal{O}$ is any other bulk field with or without spin. For example, this immediately implies that in a theory of quantum gravity where the dynamics of gravitons at low energies is described by Einstein gravity, decay of a higher spin particle into two gravitons is not allowed.

The above condition is not sufficient to completely rule out the existence of higher spin operators. In particular, we can still have higher spin operators without violating the chaos growth bound if the higher spin operator X_ℓ does not appear in the OPE of any two identical single trace primary operators. For example, if each higher spin operator has a $\mathbb{Z}_2$ symmetry, they will be prohibited from appearing in the OPE of identical operators. However, a priori we can still have non-vanishing $\langle X_\ell X_\ell O \rangle$. In fact, the Ward identity dictates that the three-point function $\langle X_\ell X_\ell T \rangle$ must be non-zero where T is the CFT stress tensor. In this section, we will utilize the holographic null energy condition to show that $\langle X_\ell X_\ell T \rangle$ must vanish for CFTs (in $D \geq 3$) with large N and a sparse spectrum, or else causality (the chaos sign bound) will be violated. The Ward identity then requires that the two-point function $\langle X_\ell X_\ell \rangle$ must vanish as well. However, the two-point function $\langle X_\ell X_\ell \rangle$ is a measure of the norm of a state created by acting X_ℓ on the vacuum and therefore must be strictly positive in a unitary CFT. Vanishing of the

[25] In terms of the conformal cross-ratios, $z \sim \sigma$ and $\bar{z} \sim \eta\sigma$. The Regge limit is defined as $\sigma \to 0$ with $\eta =$ fixed after we analytically continue $\bar{z}$ around the singularity at 1 (see [9,62,131]).

norm necessarily requires that the operator X_ℓ itself is zero.

In the gravity language, this forbids the bulk interaction $\mathcal{X}_\ell$-$\mathcal{X}_\ell$-graviton – which directly contradicts the equivalence principle. Therefore, a finite number of higher spin elementary particles, massless or massive, cannot interact with gravity in a consistent way even in AdS spacetime (in $D \geq 4$).

4.3.1 Causality and Conformal Regge Theory

We start with a general discussion about the Regge limit in generic CFTs and then review the holographic null energy condition (HNEC) in holographic CFTs which we will use to rule out higher spin single trace primary operators. The HNEC was derived in [62,131], however, let us provide a more general discussion of the HNEC here. The advantage of the new approach is that it can be applied to more general CFTs. However, that makes this subsection more technical, so casual readers can safely skip this subsection.

As discussed in [18,20,131] the relevant kinematic regime of the CFT 4-point function for accessing the physics of deep inside the bulk interior is the Regge limit. In terms of the familiar cross-ratios, in our conventions this limit corresponds to analytically continuing $\bar{z}$ around the singularity at 1 followed by taking the limit $z, \bar{z} \to 0$ with $z/\bar{z}$ held fixed. Unlike the more familiar euclidean OPE limit, the contributions to the correlation function in this limit are not easily organized in terms of local CFT operators. In fact contributions of individual local operators become increasingly singular with increasing spin. Using conformal Regge theory [32], these contributions may be resummed into finite contributions by rewriting the sum over spins as a contour integral using the Sommerfeld-Watson transform. This formalism relied on the fact that the coefficients in the conformal block expansion are well defined analytic functions of J away from integer values which was later justified in [42]. This allows one to rewrite the sum over spins in the conformal block expansion as a deformed contour integral over J, reorganizing the contributions to a sum over Regge trajectories. We will not discuss the derivation here as the details are well reviewed in [25, 26, 32, 62]. We will instead derive an expression for the contribution of a Regge trajectory directly to the OPE of two local operators in terms of a non-local operator $\mathbb{E}_{\Delta,J}$ described below.

We will first derive an expression for the contribution to the OPE of scalar operators $\psi\psi$ by an operator of spin J and scaling dimension Δ. To this end, we will utilize the methods introduced in [60] to encode primary symmetric traceless tensor operators into polynomials of degree J by contracting them with null polarization vectors z^μ :

$$\mathcal{O}(x;z) \equiv z^{\mu_1}...z^{\mu_J}\mathcal{O}(x)_{\mu_1...\mu_J}. \tag{4.3.1}$$

It was shown in [60] that the tensor may be recovered from this polynomial by using the Thomas/Todorov operator. We are however interested in the case where the spin J is not necessarily an integer. Therefore, we will employ the procedure introduced in [89] to generalize this expression to continuous spin by dropping the requirement that $\mathcal{O}(x;z)$ be a polynomial in z. With this definition, the expression for the contribution to the OPE by a continuous spin operators is given by a simple generalization of the expression appearing in [62]. We will then use the shadow representation [149–151] for the OPE in Lorentzian signature [39, 40]:

$$\left.\frac{\psi(x_1)\psi(x_2)}{\langle\psi(x_1)\psi(x_2)\rangle}\right|_{\Delta,J} = \mathcal{N}\int_{\diamond_{12}} d^d x_3 \int D^{d-2}z D^{d-2}z' \\ \times \frac{(-2z.z')^{2-d-J}\langle\psi(x_1)\psi(x_2)\tilde{\mathcal{O}}(x_3;z)\rangle}{\langle\psi(x_1)\psi(x_2)\rangle}\mathcal{O}(x_3,z'). \quad (4.3.2)$$

where we let points x_1 and x_2 to be time-like separated and the integration of x_3 is performed over the intersection of causal future of x_1 and the causal past of x_2, $\mathcal{N}$ is a normalization constant and

$$D^{d-2}z \equiv \frac{d^d z\,\delta(z^2)\theta(z_0)}{\mathrm{vol}\,\mathbb{R}_+}\ . \quad (4.3.3)$$

The integrals over z and z' replace the contraction over tensor indices that would appear for integer J using the inner product for Lorentzian principal series introduced in [89]. These are manifestly conformal integrals and the integration can be performed using the methods described in [150].

In order to obtain the contribution to the Regge limit we will set $x_1 = -x_2 = (u, v, \vec{0})$ and analytically continue the points to space-like separations resulting in integration over a complexified Lorentzian diamond. We will then take the Regge limit by sending $v \to 0$ and $u \to \infty$ with uv held fixed. The resulting expression is an integral over a complexified ball times a null ray along the u direction:

$$\left.\frac{\psi(u,v,\vec{0})\psi(-u,-v,\vec{0})}{\langle\psi(u,v,\vec{0})\psi(-u,-v,\vec{0})\rangle}\right|_{\Delta,J} = (-1)^{\frac{\Delta-1}{2}}\pi^{\frac{1-d}{2}}2^{\Delta}\frac{\Gamma\left(\frac{\Delta+J+1}{2}\right)\Gamma(\Delta-d/2+1)}{\Gamma\left(\frac{\Delta+J}{2}\right)\Gamma(\Delta-d+2)}\frac{C_{\psi\psi\mathcal{O}_{\Delta,J}}}{C_{\mathcal{O}_{\Delta,J}}} \\ \times \frac{(uv)^{\frac{d-\Delta-J}{2}}}{u^{1-J}}\int_{-\infty}^{\infty}d\tilde{u}\int_{\vec{x}^2\le uv} d^{d-2}\vec{x}(uv-\vec{x}^2)^{\Delta-d+1}\mathcal{O}((\tilde{u},0,i\vec{x});(0,1,0)) \\ \equiv u^{J-1}\mathbb{E}_{\Delta,J}\ , \quad (4.3.4)$$

where $C_{\psi\psi\mathcal{O}_{\Delta,J}}$ is the OPE coefficient, $C_{\mathcal{O}_{\Delta,J}}$ is the normalization of $\langle\mathcal{O}\mathcal{O}\rangle$ and we have used $(u, v, \vec{x}_\perp)$ to express coordinates. This operator captures the contribution to OPE of $\psi\psi$ in the Regge limit. Therefore, analytically continued conformal blocks can be computed by inserting $\mathbb{E}_{\Delta,J}$ inside a three-point function. For example, in the case of external scalars we find

$$\begin{aligned}\frac{\langle\phi(x_3)\phi(x_4)\mathbb{E}_{\Delta,J}\rangle}{\langle\phi(x_3)\phi(x_4)\rangle}u^{J-1} &\sim \lim_{\substack{z,\bar{z}\to 0\\ z/\bar{z}\text{ fixed}}} G^{\circlearrowleft}_{\Delta,J}(z,\bar{z})\\ &= \frac{i(-1)^J 2^{2\Delta+3J-2}\Gamma\left(\frac{J+\Delta-1}{2}\right)\Gamma\left(\frac{J+\Delta+1}{2}\right)}{\Gamma\left(\frac{J+\Delta}{2}\right)^2}\frac{z^{\frac{\Delta-J}{2}}\bar{z}^{-\frac{\Delta}{2}-\frac{J}{2}+2}}{(z-\bar{z})}\ ,\end{aligned} \tag{4.3.5}$$

where $G^{\circlearrowleft}_{\Delta,J}(z,\bar{z})$ is obtained from the conformal block by taking $\bar{z}$ around 1 while holding z fixed. In (4.3.4) this analytic continuation corresponds to the choice of contour in performing the $\tilde{u}$ integral. The integrand encounters singularities in $\tilde{u}$ as the points become null separated from x_3 or x_4. Different analytic continuations of the conformal block can be obtained by choosing appropriate contours. The choice of contour in the $\tilde{u}$ plane was discussed in [131] in greater detail. By an identical Sommerfeld-Watson transform and contour deformation argument as in [32], the expression for the Regge OPE can now be used to capture the contribution of Regge trajectories

$$\left.\frac{\psi(u,v,\vec{0})\psi(-u,-v,\vec{0})}{\langle\psi(u,v,\vec{0})\psi(-u,-v,\vec{0})\rangle}\right|_{J(\nu)} = \int d\nu\, u^{J(\nu)-1}a(\nu)\mathbb{E}_{\Delta(J(\nu)),J(\nu)}\ , \tag{4.3.6}$$

where the coefficient $a(\nu)$ encodes the dynamical information about the spectrum of the CFT for the Regge trajectory parametrized by $J(\nu)$.

The operator $\mathbb{E}_{\Delta,J}$ can be contrasted with the light-ray operator $\mathbf{L}[\mathcal{O}]$ introduced in [89]. Although both correspond to non-local contributions to the OPE in the Regge limit, they do not compute the same quantity. As mentioned above $\mathbb{E}_{\Delta,J}$ computes the analytic continuation of the conformal block, whereas $\mathbf{L}[\mathcal{O}]$ computes the analytic continuation of conformal partial wave which is the sum of the block and its shadow which is proportional to $G_{1-J,1-\Delta}(z,\bar{z})$. However, because of the symmetry of the coefficient $a(\nu)$ under $\nu \to -\nu$ using either operator in the Regge limit will yield the same results after integration.

Holographic CFT: Holographic Null Energy Condition

As described in more detail in [25,26,29,32,62,66] the leading Regge trajectory in a holographic theory with a large Δ_{gap} can be parametrized as

$$J(\nu) = 2 - \frac{1}{\Delta_{gap}^2}\left(\frac{d^2}{4} + \nu^2\right) + \mathcal{O}\left(\frac{1}{\Delta_{gap}^4}\right) \ . \tag{4.3.7}$$

Using this expression for the trajectory we find that at leading order in Δ_{gap} the coefficient $a(\nu)$ will have single poles corresponding to the stress-tensor exchange as well as an infinite set of double-trace operators. As shown in [62,131], in the class of states in which we are interested, the dominant contribution to this OPE is given by the stress-tensor and the double-trace operators will not contribute. This contribution is captured by the holographic null energy operator

$$\mathbf{E}_r(v) = \int_{-\infty}^{+\infty} du' \int_{\vec{x}^2 \leq r^2} d^{d-2}\vec{x}\left(1 - \frac{\vec{x}^2}{r^2}\right) T_{uu}\left(u', v, i\vec{x}\right) \tag{4.3.8}$$

which is a generalization of the averaged null energy operator [62] and a special case of the operator $\mathbb{E}_{\Delta,J}$ described above with $\Delta = d$ and $J = 2$.[26] In particular, in the limit $r \to 0$, this operator is equivalent to the averaged null energy operator.

Causality in CFT implies that the four-point function obeys certain analyticity properties [10,12,63,64]. For generic CFTs in $D \geq 3$, these analyticity conditions

[26] We are using the following convention for points $x \in \mathbb{R}^{1,d-1}$ in CFT$_d$:

$$x = (t, x^1, \vec{x}) \equiv (u, v, \vec{x}) \ , \qquad \text{where,} \qquad u = t - x^1 \ , \qquad v = t + x^1 \ . \tag{4.3.9}$$

dictate that the averaged null energy operator must be non-negative [10]. However, for holographic CFTs, causality leads to stronger constraints. In particular, causality of CFT four-point functions in the Regge limit implies that the expectation value of the holographic null energy operator is positive in a subspace of the total Hilbert space of holographic CFTs [62, 131]:

$$\mathbf{E}(\rho) \equiv \lim_{B\to\infty} \langle\Psi|\mathbf{E}_{\sqrt{\rho}B}(B)|\Psi\rangle \geq 0 \ , \tag{4.3.10}$$

where, $0 < \rho < 1$. The class of states $|\Psi\rangle$ are created by inserting an arbitrary operator O near the origin

$$|\Psi\rangle = \int dy^1 d^{d-2}\vec{y}\, \epsilon.O(-i\delta, y^1, \vec{y})|0\rangle \ , \qquad \langle\Psi| = \int dy^1 d^{d-2}\vec{y}\, \langle 0|\epsilon^*.O(i\delta, y^1, \vec{y}) \ , \tag{4.3.11}$$

where, ϵ is the polarization of the operator O with

$$\epsilon.O \equiv \epsilon_{\mu\nu\cdots} O^{\mu\nu\cdots} \tag{4.3.12}$$

and $\delta > 0$. The state $|\Psi\rangle$ is equivalent to the Hofman-Maldacena state of the original conformal collider [13] which was created by acting local operators, smeared with Gaussian wave-packets, on the CFT vacuum.

The HNEC is practically a conformal collider experiment for holographic CFTs (in $D \geq 3$) in which the CFT is prepared in an excited state $|\Psi\rangle$ by inserting an operator O near the origin and an instrument measures $\mathbf{E}(\rho)$ far away from the excitation, as shown in figure 4.6. Then, causality implies that the measured value $\mathbf{E}(\rho)$ must be non-negative for large-N CFTs with a sparse spectrum. Next, creating the state $|\Psi\rangle$ by inserting the higher spin operator X_ℓ, we show that the inequality (4.3.10) leads to surprising equalities among various OPE coefficients that appear in $\langle X_\ell X_\ell T\rangle$.

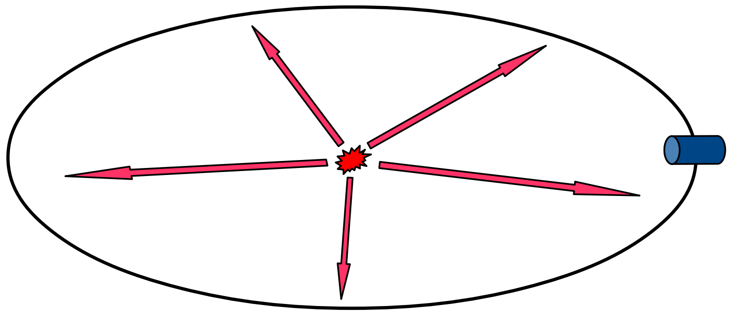


Figure 4.6: Holographic null energy condition (HNEC): A holographic CFT is prepared in an excited state $|\Psi\rangle$ by inserting an operator O near the origin and an instrument which is shown in blue, measures the holographic null energy $\mathbf{E}_r$ far away from the excitation.

4.3.2 $D > 4$

We will use the HNEC to derive bounds on higher spin single trace primary operators in $D \geq 4$ (or AdS$_D$ with $D \geq 5$). We will explicitly show that spin 3 and 4 operators are completely ruled out and then argue that the same must be true even for $J > 4$. The case of $D = 4$ is more subtle and will be discussed separately.

Spin-3 Operators

Let us start with an operator X_ℓ with $\ell = 3$ which does not violate the chaos growth bound because it has $\mathbb{Z}_2$ or some other symmetry which sets $\langle OOX_{J=3}\rangle = 0$ for all O. Consequently, this operator does not contribute as an exchange operator in any four-point function in the Regge limit and the leading contribution to the Regge four-point function still comes from the exchange of spin-2 single trace (stress tensor) and double trace operators. Therefore, the HNEC is still valid and we can use it with states created by smeared $X_{\ell=3}$ to derive constraints on $\langle X_{\ell=3}X_{\ell=3}T\rangle$.

The CFT three-point function $\langle X_{\ell=3}X_{\ell=3}T\rangle$, is completely fixed by conformal symmetry up to a finite number of OPE coefficients (see appendix A.13). After imposing permutation symmetry and conservation equation, the three-point function $\langle X_{\ell=3}X_{\ell=3}T\rangle$ has 9 independent OPE coefficients. We now compute the expectation value of the holographic null energy operator $\mathbf{E}(\rho)$ in states created by smeared $X_{\ell=3}$:

$$|\Psi\rangle = \int dy^1 d^{d-2}\vec{y}\,\epsilon^{\mu_1}\epsilon^{\mu_2}\epsilon^{\mu_3}X_{\mu_1\mu_2\mu_3}(-i\delta, y^1, \vec{y})|0\rangle\ , \tag{4.3.13}$$

where, ϵ^μ is a null polarization vector:

$$\epsilon^\mu = (-i\xi, -i, \vec{\varepsilon}_\perp)\ , \tag{4.3.14}$$

with $\xi = \pm 1$ and $\vec{\varepsilon}_\perp{}^2 = 0$.[27] Following the procedure outlined in [131], we can compute $\mathbf{E}(\rho)$ in state (4.3.13). The result has the following form

$$\mathbf{E}(\rho) = \frac{1}{(1-\rho)^{d+3}}\sum_{n=0}^{\infty} I_\xi^{(n)}(\lambda^2)(1-\rho)^n\ , \tag{4.3.15}$$

where, $I_\xi^{(n)}(\lambda^2)$ are polynomials in λ^2 which in general have terms up to order λ^6, where

$$\lambda^2 = \frac{1}{2}\vec{\varepsilon}_\perp\cdot\vec{\varepsilon}^{\,*}_\perp \geq 0\ . \tag{4.3.16}$$

Given our choice of polarization, different powers of λ^2 correspond to independent spinning structures and decomposition of $SO(d-1,1)^3$ to representations under $SO(d-2)$. Therefore, positivity of $\mathbf{E}(\rho)$ implies that the coefficients of each power of λ^2 must individually satisfy positivity, for $\xi = +1$ as well as $\xi = -1$. Now, applying the HNEC order by order in the limit $\rho \to 1$, the inequalities lead to 9 equalities among the 9 OPE coefficients. We find that the 9 OPE coefficients

[27] Note that in $D = 3$ this choice of polarization vector does not work. In this case, one needs to use a general polarization tensor to derive constraints.

cannot be consistently chosen to satisfy these equalities. Hence, causality implies that

$$\langle X_{\ell=3} X_{\ell=3} T \rangle = 0 \ . \tag{4.3.17}$$

Moreover, the Ward identity relates C_{X_3}, coefficient of the two-point function $\langle X_{\ell=3} X_{\ell=3} \rangle$ (see eq A.13.2), to a particular linear combination of the OPE coefficients $C_{i,j,k}$ and hence the two-point function $\langle X_{\ell=3} X_{\ell=3} \rangle$ must vanish as well. This implies that we cannot have individual spin-3 single trace primary operators in the spectrum. The detail of the calculation are rather long and not very illuminating, so we relegate them to appendix A.14.

Spin-4 Operators

We can perform a similar analysis with a spin-4 operator which leads to the same conclusion, however, the details are little different. The three-point function $\langle X_{\ell=4} X_{\ell=4} T \rangle$, after imposing permutation symmetry and conservation equation, has 12 independent OPE coefficients (see appendix A.15). But the HNEC leads to stronger constraints as we increase the spin of X and these 12 OPE coefficients cannot be consistently chosen to satisfy all the positivity constraints. In fact, as we will show, it is easier to rule out spin-4 operators using the HNEC than spin-3 operators.

We again perform a conformal collider experiment for holographic CFTs (in $D \geq 3$) in which the CFT is prepared in an excited state

$$|\Psi\rangle = \int dy^1 d^{d-2}\vec{y}\, \epsilon^{\mu_1}\epsilon^{\mu_2}\epsilon^{\mu_3}\epsilon^{\mu_4} X_{\mu_1\mu_2\mu_3\mu_4}(-i\delta, y^1, \vec{y})|0\rangle \ , \tag{4.3.18}$$

where, ϵ^μ is the null polarization vector (4.3.14). The expectation value of the holographic null energy operator $\mathbf{E}(\rho)$ in states created by smeared $X_{\ell=4}$ can be computed using methods used in [131]

$$\mathbf{E}(\rho) = \frac{1}{(1-\rho)^{d+5}} \sum_{n=0}^{\infty} \tilde{I}_{\xi}^{(n)}(\lambda)(1-\rho)^n \ , \tag{4.3.19}$$

where, $\tilde{I}_{\xi}^{(n)}(\lambda^2)$ are polynomials in λ^2 (4.3.16) with terms up to λ^8 in general. Causality implies that different powers of λ^2 must satisfy positivity individually, for $\xi = +1$ as well as $\xi = -1$. We find that the 12 OPE coefficients cannot be consistently chosen to satisfy all the positivity constraints implying (see appendix A.15)

$$\langle X_{\ell=4} X_{\ell=4} T \rangle = 0 \ . \tag{4.3.20}$$

Consequently, the Ward identity dictates that the two-point function of $X_{\ell=4}$ must vanish as well. This rules out single trace spin-4 operators with scaling dimensions below Δ_{gap} in the spectrum of a holographic CFT. As shown in the appendix A.15, we ruled out spin-4 operators even without considering $\mathbf{E}_{\xi=-1}(\rho)$. This is because as we increase the spin of X, the number of constraint equations increases faster than the number of independent OPE coefficients. This is also apparent from the fact that for spin-3, we had to go to order $\frac{1}{(1-\rho)^{d-2}}$ to derive all constraints. Whereas, for spin-4, the full set of constraints were obtained at the order $\frac{1}{(1-\rho)^{d-1}}$.

Spin $\ell > 4$

For operators with spin $\ell \geq 5$, the argument is exactly the same. In fact, it is easier to rule them out because the HNEC leads to stronger constraints at higher spins. For example, for $\ell = 1$, there are 3 independent OPE coefficients but the HNEC yields 2 linear relations among them. Consequently, the three-point function $\langle X_{\ell=1} X_{\ell=1} T \rangle$ is fixed up to one coefficient. The same is true for

$\ell = 2$ – there are 6 independent OPE coefficients and 5 constraints from the HNEC. Furthermore, in both of these cases, constraint equations ensure that the expectation value of the holographic null energy operator behaves exactly like that of the scalars: $\mathbf{E}(\rho) \sim \frac{1}{(1-\rho)^{d-3}}$ for $D \geq 4$. In fact, this is true for all low spin operators of holographic CFTs.

The HNEC barely rules out operators with $\ell = 3$. There are 9 independent OPE coefficients. Using the positivity conditions all the way up to order $\frac{1}{(1-\rho)^{d-2}}$ for $\xi = \pm 1$, we showed that the OPE coefficients cannot be consistently chosen to satisfy all the positivity constraints. Whereas, the HNEC rules out $\ell = 4$ operators quite comfortably. We only needed to consider positivity conditions up to order $\frac{1}{(1-\rho)^{d-1}}$ and only for $\xi = +1$ to rule them out. The same pattern persists even for operators with spins $\ell \geq 5$ so we will not repeat our argument for each spin. Instead, we present a general discussion about the structure of $\mathbf{E}(\rho)$ at each order in the limit $\rho \to 1$ for general Δ and J (in $D \geq 4$ dimensions). This enables us to count the number of constraint equations at each order. A simple counting immediately suggests that a non-vanishing $\langle X_\ell X_\ell T \rangle$ cannot be consistent with the HNEC even for spins higher than 4. By studying various examples with specific values of ℓ, Δ and D, we have explicitly checked that our simple counting argument is indeed true.

The three point function $\langle X_\ell X_\ell T \rangle$ has $5 + 6(\ell - 1)$ OPE coefficients to begin with, however not all of them are independent. Permutation symmetry implies that only 4ℓ OPE coefficients can be independent. In addition, conservation of the stress-tensor operator T imposes ℓ additional constraints among the remaining 4ℓ OPE coefficients. Therefore, the three-point function $\langle X_\ell X_\ell T \rangle$ is fixed by conformal invariance up to 3ℓ truly independent OPE coefficients.[28] Furthermore, the Ward identity leads to a relation between these OPE coefficients and the coefficient of the two-point function C_{X_ℓ}.

We again perform a conformal collider experiment for holographic CFTs (in $D \geq 4$) in which the CFT is prepared in an excited state created by smeared X_ℓ. In the limit $\rho \to 1$, the leading contribution to $\mathbf{E}(\rho)$ goes as

$$\mathbf{E}(\rho) \sim \frac{1}{(1-\rho)^{d+2\ell-3}} , \tag{4.3.21}$$

where only a single structure contributes with an overall factor that depends on a specific linear combination of OPE coefficients. Just like before, the structure changes sign for different powers of λ^2 and hence in the 1st order, the HNEC produces only one constraint. It is clear from [62, 131] that the coefficient of the term $\mathbf{E}(\rho) \sim \frac{1}{(1-\rho)^{d-3}}$ is fixed by the Ward identity and hence automatically positive. On the other hand, the HNEC in general can lead to constraints up to the 2ℓ-th order, i.e. the order $\mathbf{E}(\rho) \sim \frac{1}{(1-\rho)^{d-2}}$. But for $\ell > 3$, one gets 3ℓ independent constraints from the HNEC even before the 2ℓ-th order.

It is easier to rule out operators with higher and higher spins. A simple counting clearly shows why this is not at all surprising. First, let us assume that the HNEC rules out any operator with some particular spin $\ell = \ell_* > 2$. That means for spin ℓ_* the HNEC generates $3\ell_*$ independent relations among the OPE coefficients. If we increase the spin by 1: $\ell = \ell_* + 1$, we get 3 more independent OPE coefficients. However, the $(2\ell_* + 1)$-th and $(2\ell_* + 2)$-th orders in $\mathbf{E}(\rho)$ produce new constraints and at each new order there can be $\ell_* + 1$ new equalities. Moreover, the λ^2 polynomials at each order now has a $\lambda^{2(\ell_*+1)}$ term with its own positivity condition

[28]The number of independent OPE coefficients is different in $D = 3$.

– this means that there can be $2\ell_*$ additional equalities from the first $2\ell_*$ orders. Therefore, for spin $\ell_* + 1$, there are 3 new OPE coefficients, whereas there can be $2(2\ell_* + 1)$ new constraints among them. Of course, this is not exactly true because some of $2(2\ell_* + 1)$ constraints are not independent. However, for $\ell_* \geq 4$, the number of new constraints $2(2\ell_* + 1) \gg 3$ and hence this simple counting suggests that the HNEC must rule out operators with spin $\ell \geq 5$.

Let us now demonstrate that this simple counting argument is indeed correct. First, consider $\ell = 1$. This is the simplest possible case which was studied in [131]. For $\ell = 1$, there are 3 independent OPE coefficients. The number of constraints (equality) from the HNEC at each order is given by $\{1, 1\}$.[29] After imposing these constraints the expectation value of the holographic null energy operator goes as $\sim \frac{1}{(1-\rho)^{d-3}}$. Similarly, for $\ell = 2$ the number of constraints from the HNEC at each order is given by {1,1,2,1} and the total number of constraints is still less than the number of independent OPE coefficients [131].

For $\ell = 3$, the sequence is $\{1, 1, 2, 2, 2, 1\}$ (see appendix A.14) and hence spin-3 operators were completely ruled out at the order $\frac{1}{(1-\rho)^{d-2}}$. If we increase the spin by 1, we find that the number of constraints from the HNEC at each order is $\{1, 1, 2, 2, 3, 2, 1, 0\}$ (see appendix A.15). The zero at the end indicates that spin-4 operators were already ruled out at the order $\frac{1}{(1-\rho)^{d-1}}$. Our simple counting suggests that the number of zeroes should increase as we go to higher spins. Explicit computation agrees with this expectation. In particular, for $\ell = 5$, there are 15 independent OPE coefficients and the number of constraints at each order is $\{1, 1, 3, 3, 5, 2, 0, 0, 0, 0\}$. Therefore, the spin-5 operators are ruled out at

[29]The n-th element of the sequence $\{c_1, \cdots, c_n, \cdots, c_{2\ell}\}$ represents the number of independent constraints at the order n.

the order $\frac{1}{(1-\rho)^{d+2}}$. Similarly, for $\ell = 6$, there are 18 independent OPE coefficients. Explicit calculation shows that the number of constraints at each order is $\{1, 1, 3, 3, 5, 5, 0, 0, 0, 0, 0, 0\}$. Therefore, spin-6 operators can be ruled out even at the order $\frac{1}{(1-\rho)^{d+4}}$. All of these results imply that the presence of any single trace primary operator with spin $\ell > 2$ is not compatible with causality.

4.3.3 AdS$_4$/CFT$_3$

Similar to the $D = 4$ case on the gravity side, CFTs in $D = 3$ are special. Of course, large-N CFTs with a sparse spectrum in $(2+1)$-dimensions are still holographic and the HNEC once again implies that higher spin single trace operators with $\Delta \ll \Delta_{gap}$ are ruled out. However, there are several aspects of the $D = 3$ CFTs which are different from the higher dimensional case.

First of all, in CFT$_3$ the three-point functions $\langle X_\ell X_\ell T\rangle$ have both parity even and parity odd structures for any ℓ

$$\langle X_\ell X_\ell T\rangle = \langle X_\ell X_\ell T\rangle_+ + \langle X_\ell X_\ell T\rangle_- \ . \tag{4.3.22}$$

Furthermore, the number of independent parity even structures at $D = 3$ is different from the higher dimensional case. The general three-point function (A.13.4) implies that after imposing permutation symmetry and conservation equation, similar to the higher dimensional case $\langle X_\ell X_\ell T\rangle_+$ should contain 3ℓ independent structures. However, for $D = 3$, not all of these structures are independent. In particular, this overcounting should be corrected by setting OPE coefficients $C_{1,1,k} = 0$ for $k \geq 1$ in (A.13.4) [60]. Therefore, in $D = 3$, the parity even part $\langle X_\ell X_\ell T\rangle_+$ has $2\ell + 1$ independent OPE coefficients. Whereas, the parity odd part $\langle X_\ell X_\ell T\rangle_-$

has 2ℓ independent OPE coefficients. Note that this is exactly what is expected from interactions of gravitons with higher spin fields in 4d gravity.

There is another aspect of $D = 3$ which is different from the higher dimensional case. The choice of polarization (4.3.14) in $D = 3$ implies that $\vec{\varepsilon}_\perp = 0$ and hence the λ-trick does not work. However, the full set of bounds can be obtained by considering the full polarization tensor for X_ℓ. This can be achieved by using the projection operator of [60] which makes the analysis more complicated. However, the final conclusion remains unchanged.

Since we expect that the HNEC imposes stronger constraints as we increase the spin, it is sufficient to only rule out $X_{\ell=3}$. The steps are exactly the same but details are little different. After imposing permutation symmetry and conservation equation, the three-point function $\langle X_{\ell=3} X_{\ell=3} T\rangle$ has 7 parity even and 6 parity odd independent OPE coefficients. We again compute the expectation value of the holographic null energy operator $\mathbf{E}(\rho)$ in states created by smeared $X_{\ell=3}$:

$$|\Psi\rangle = \int dy^1 dy^2\, \epsilon^{\mu_1\mu_2\mu_3} X_{\mu_1\mu_2\mu_3}(-i\delta, y^1, y^2)|0\rangle \ , \tag{4.3.23}$$

where $\epsilon^{\mu_1\mu_2\mu_3}$ is the traceless symmetric polarization tensor. Using the techniques developed in [131], we now compute the expectation value of the holographic null energy operator $\mathbf{E}(\rho)$ in this state which can be schematically expressed in the following form

$$\mathbf{E}(\rho) = \sum_{n=1}^{6} \frac{j_n(\epsilon^{\mu_1\mu_2\mu_3}, C_{i,j,k})}{(1-\rho)^n} + j_0(\epsilon^{\mu_1\mu_2\mu_3}, C_{i,j,k}) \ln(1-\rho) + \cdots \ , \tag{4.3.24}$$

where $j_n(\epsilon^{\mu_1\mu_2\mu_3}, C_{i,j,k})$ are specific functions of the the polarization tensors and the OPE coefficients. The dots in the above expression represent terms that vanish in the limit $\rho \to 1$. The $\ln(1-\rho)$ term is unique to the 3d case and is a manifestation of soft graviton effects in the IR.

By applying the HNEC order by order in the limit $\rho \to 1$, we again find that the HNEC can only be satisfied for all polarizations if and only if all the OPE coefficients vanish. Consequently, the Ward identity implies that we cannot have individual spin-3 operators in the spectrum.[30] Moreover, a simple counting again suggests that the same is true even for $\ell > 3$. In $D = 3$, as we increase the spin by one, the number of parity even OPE coefficients increases by 2. However, now there are two more orders perturbatively in $(1 - \rho)$ that generate new relations among the OPE coefficients. Each new order produces at least one new constraint suggesting that if the HNEC rules out parity even operators with some particular spin ℓ, it will also rule out all parity even operators with spin $\ell+1$. In addition, it is straightforward to extend this argument to include parity odd structures, however, we will not do so in this chapter.

4.3.4 Maldacena-Zhiboedov Theorem and Massless Higher Spin Fields

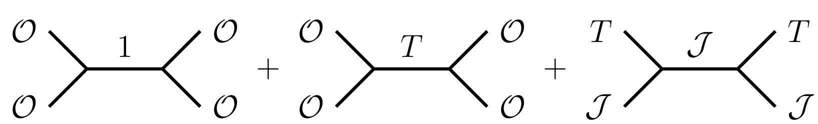
In this section we argued that in holographic CFTs, any higher spin single trace non-conserved primary operator violates causality. On the gravity side, this rules out any higher spin massive field with mass below the cut-off scale (for example the string scale). But what about massless higher spin fields? In asymptotically flat spacetime, this question has already been answered by the Weinberg-

[30] As explained in appendix A.16 it is still possible to use the λ-trick to derive constraints in dimension $D = 3$. This implies that individual spin-4 single trace operators (at least the parity even part) are also ruled out.

Witten/Porrati theorem [113, 114]. The same statement can be proven in AdS by using the argument of this section but for conserved $X_\ell \equiv \mathcal{J}$. Conservation of $\mathcal{J}$ leads to additional relations among the OPE coefficients $C_{i,j,k}$'s in $\langle \mathcal{J}\mathcal{J}T\rangle$. Even before we impose these additional conservation relations, the HNEC implies $C_{i,j,k} = 0$ for $\ell > 2$, which is obviously consistent with these new relations from conservation. Hence, our argument is valid even for higher spin conserved current $\mathcal{J}$.

Causality of CFT four-point functions in the lightcone limit also rules out a finite number of conserved higher spin currents in any CFT [63]. This is a partial generalization of the Maldacena-Zhiboedov theorem [132], from $D = 3$ to higher dimensions. The argument which was used in [63] to rule out higher spin conserved current is not applicable here since $\mathcal{J}$ does not contribute to generic CFT four-point functions as exchange operators.[31] However, we can repeat the argument of [63] for a mixed correlator $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ in the lightcone limit where, $\mathcal{O} \equiv T + \mathcal{J}$. For this mixed correlator, $\mathcal{J}$ does contribute as an exchange operator in the lightcone limit. In particular, we can schematically write

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle =$$



$$+\cdots\ , \tag{4.3.25}$$

where each diagram represents a spinning conformal block and dots represent contributions suppressed by the lightcone limit. The argument of [63], now applied to the correlator $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$, implies that this correlator is causal if and only if the last term in (4.3.25) is identically zero. The $\mathcal{J}$-exchange conformal blocks, for $\ell > 2$, in the lightcone grow faster than allowed by causality. This necessarily

[31] Let us recall that none of the operators are charged under $\mathcal{J}$ and hence one can tune $\langle \mathcal{J}OO\rangle = 0$ for any O. Consequently, $\mathcal{J}$ does not contribute as an exchange operator.

requires that the three-point function $\langle \mathcal{J}\mathcal{J}T\rangle$ must vanish – which is sufficient to rule out $\mathcal{J}$ for $\ell > 2$. This generalizes the argument of [63] ruling out higher spin conserved currents even when none of the operators are charged under it. We should note that technically it might be plausible for the OPE coefficients to conspire in a non-trivial way such that a conserved current $\mathcal{J}$ cannot contribute as an exchange operator (for all polarizations of the external operators) but still has a non-vanishing $\langle \mathcal{J}\mathcal{J}T\rangle$. However, it is very unlikely that such a cancellation is possible since the three-point function $\langle \mathcal{J}\mathcal{J}T\rangle$ can only have three independent OPE coefficients. This unlikely scenario can be ruled out by explicit calculations.

The above argument is applicable only because $\mathcal{J}$ is conserved. However, one might expect that a similar argument in the Regge limit should rule out even non-conserved X_ℓ for holographic CFTs. This is probably true but the argument is more subtle in the Regge limit because an infinite tower of double trace operators also contribute to the correlator $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$. Hence, one needs to smear all four operators appropriately, in a way similar to [9, 62], such that the double trace contributions are projected out. One might then use causality/chaos bounds to rule out the three-point function $\langle X_\ell X_\ell T\rangle$. However, it is possible that the smearing procedure sets contributions from certain spinning structures in $\langle X_\ell X_\ell T\rangle$ to zero as well. In that case, this argument will not be sufficient. A proof along this line requires the computation of a completely smeared spinning Regge correlator which is technically challenging even in the holographic limit.

4.3.5 Comments

Small Deviation from the Holographic Conditions

Large-N CFTs with a sparse spectrum are indeed special because at low energies they exhibit gravity-like behavior. This immediately poses a question about the assumptions of large-N and sparse spectrum: how rigid are these conditions? In other words, do we still get a consistent CFT if we allow small deviations away from these conditions?

In this section, we answered a version of this question for the sparseness condition. The sparseness condition requires that any single trace primary operator with spin $\ell > 2$, must necessarily have dimension $\Delta \geq \Delta_{gap} \gg 1$. This condition ensures that the dual gravity theory has a low energy description given by Einstein gravity. However, we can imagine a small deviation from this condition by allowing a finite number of additional higher spin single trace primary operators X_ℓ with $\ell > 2$ and scaling dimension $\Delta \ll \Delta_{\text{gap}}$. As we have shown in this section, these new operators violate the HNEC implying the resulting CFTs are acausal.

Minkowski vs AdS

It is rather apparent that the technical details of the flat spacetime argument and the AdS argument are very similar. For example, the number of independent structures for a particular spin is the same in both cases. In flat spacetime as well as in AdS, we start with inequalities which can be interpreted as some kind of time-delay. In addition, these inequalities when applied order by order, lead to equalities among various structures. These equalities eventually rule out higher spin particles. However, the AdS argument has one conceptual advantage, namely, it does not require any additional assumption about the exponentiation of the leading contribution. The CFT-based argument relies on the HNEC. The derivation of the HNEC utilized the causality of a CFT correlator which was designed to probe high energy scattering deep into the AdS bulk. It is therefore not a coincidence that the technical details of the AdS and the flat space arguments are so similar. Since the local high energy scattering is insensitive to the spacetime curvature, it is not very surprising that the bounds in flat space and in AdS are identical. This also suggests that the same bound should hold even in de Sitter.

Higher Spin Operators in Generic CFTs

The argument of this section does not rule out higher spin non-conserved operators in non-holographic CFTs. However, the HNEC in certain limits can be utilized to constrain interactions of higher spin operators even in generic CFTs. In particular, the limit $\rho \to 0$ in (4.3.10) corresponds to the lightcone limit and in this limit, the HNEC becomes the averaged null energy condition (ANEC). The proof of the ANEC [10,46] implies that in the limit $\rho \to 0$, the inequality $\mathbf{E}(\rho) \geq 0$ must be true for any interacting CFT in $D \geq 3$. Moreover, in this limit, the HNEC is equivalent to the conformal collider setup of [13] which is known to yield optimal bounds. Therefore, the same computation performed in the limit $\rho \to 0$ can be used to derive non-trivial but weaker constraints on the three-point functions $\langle X_\ell X_\ell T \rangle$ which are true for any interacting CFT in $D \geq 3$. These constraints, even though easy to obtain from our calculations of $\mathbf{E}(\rho)$, are rather long and complicated and we will not transcribe them here.

Other Applications of the Regge OPE

In chapter we specialized $\mathbb{E}_{\Delta,J}$ to the case of $\Delta = d$ and $J = 2$ to arrive at the HNEC operator in order to make use of the universality of the stress-tensor Regge trajectory in holographic theories. However, $\mathbb{E}_{\Delta,J}$ more generally describes the contribution of any operator to the Regge OPE of identical scalar operators. It would be interesting to find the actual spectrum of these operators contributing to the Regge limit of the OPE in specific theories. It would also be worthwhile to try and understand the subleading contributions to the Regge OPE in holographic theories. Although these contributions are not universal, we expect that causality will impose constraints on these contributions as well.

We have explored the Regge limit of the OPE of two identical scalars. Generalization to other representations is straightforward as it only requires knowledge of the CFT three-point functions whose functional form is fixed by symmetry. Positivity of these generalized Regge OPE operators will likely lead to new constraints since they allow access to more general representations. Furthermore, decomposition of the additional Lorentz indices under the little group will result on more constraint equations which need to be satisfied to preserve causality.

4.4 Restoring Causality

4.4.1 Make CFT Causal Again

In the previous section, we considered large-N CFTs in $D \geq 3$ dimensions with the property that the lightest single trace operator with spin $\ell > 2$ has dimension $\Delta \equiv \Delta_{\text{gap}} \gg 1$. These holographic conditions are equivalent to the statement that in the gravity side the low energy behavior is governed by the Einstein gravity. Moreover, Δ_{gap} corresponds to the scale of new physics Λ in the effective action in AdS (for example it can be the string scale M_s). In any sensible theory of quantum gravity it is expected that the Einstein-Hilbert action should receive higher derivative corrections which are suppressed by the scale Λ. On the CFT side, this translates into the fact that there is an infinite tower of higher spin operators with dimensions above the Δ_{gap}. All of these higher spin operators must appear as exchange operators in CFT four-point functions in order to restore causality at high energies [9]. Furthermore, in this chapter we showed that the sparseness condition is very rigid and we are not allowed to add an additional higher spin operator X_ℓ with spin $\ell > 2$ and $\Delta \ll \Delta_{\text{gap}}$ if causality is to be preserved. Let us consider adding an additional higher spin primary single trace operator X_ℓ with dimension $\Delta = \Delta_0 \ll \Delta_{\text{gap}}$ (or on the gravity side a higher spin particle with mass $M_0 \ll \Lambda$) and ask whether it is possible to restore causality by adding one or more primary operators (or new particles) that cancel the causality violating contributions? In this section, we answer this question from the CFT side.

The bound obtained in the previous section from the HNEC is expected to be exact strictly in the limit $\Delta_{\text{gap}} \to \infty$. However, it is easy to see that the same conclusion is true even when Δ_{gap} is large but finite, as long as $\Delta_0 \ll \Delta_{\text{gap}}$. In this case, one might expect that the OPE coefficients are no longer exactly zero but receive corrections $C_{i,j,k}/C_{X_\ell} \sim \frac{1}{\Delta_{\text{gap}}^a}$, where a is some positive number.[32] However, this is inconsistent with the Ward identity which requires that at least some of $C_{i,j,k}/C_{X_\ell} \sim \mathcal{O}(1)$. Therefore, even for large but finite Δ_{gap}, the operator X_ℓ is ruled out as long as $\Delta_0 \ll \Delta_{\text{gap}}$. In addition, this also implies that if we want to add X_ℓ, it will not be possible to save causality by changing the spectrum above Δ_{gap}. Let us add extra operators at dimensions $\sim \Delta'_{\text{gap}} \ll \Delta_{\text{gap}}$ in order to restore causality. Note that if $\Delta'_{\text{gap}} \gg \Delta_0$, then contributions of these extra operators are expected to be suppressed by Δ'_{gap} and hence we can again make the above argument. Therefore, contributions of these extra operators can be significant enough to restore causality if and only if $\Delta'_{\text{gap}} \sim \Delta_0$.

The above argument also implies that perturbative $1/N$ effects are not sufficient to save causality either. Any such correction must be suppressed by positive powers of $1/N$ and hence inconsistent with the Ward identity. This is also clear from the gravity side, both in flat space and in AdS. Causality requires that the tree level higher spin-higher spin-graviton amplitude must vanish. One might expect that loop effects can generate a non-vanishing amplitude without violating causality, however, these effects must be $1/N$ suppressed. Hence, this scenario is in tension with the universality of gravitational interactions dictated by the equivalence principle.

The behavior of four-point functions in the Regge limit makes it obvious that

[32] C_{X_ℓ} is the coefficient of the two-point function of X_ℓ and $C_{i,j,k}$ are the OPE coefficients for $\langle X_\ell X_\ell T\rangle$ (see appendix A.13).

these extra operators at $\Delta'_{\rm gap}$ must have spin $\ell \geq 2$ so that they can contribute significantly in the Regge limit to restore causality. Furthermore, causality imposes strong restrictions on what higher spin operators can be added at $\Delta'_{\rm gap}$. The simplest possibility is to add a finite or infinite set of higher spin operators at $\Delta'_{\rm gap}$ which do not contribute as exchange operators in any four-point functions. However, this scenario makes the causality problem even worse. The causality of the Regge four point functions still leads to the HNEC and one can rule out even an infinite set of such operators by applying the HNEC to individual higher spin operators. The only other possibility is to add a set of higher spin operators at $\Delta'_{\rm gap}$ which do contribute as exchange operators in the four-point function $\langle X_\ell X_\ell \psi\psi\rangle$, where ψ is a heavy scalar operator. In this case2, the HNEC is no longer applicable and hence the argument of the previous section breaks down. However, a finite number of higher spin primaries ($\ell > 2$) that contribute as exchange operators violate chaos/causality bound [9, 41] and consequently this scenario necessarily requires an infinite tower of higher spin operators.[33] Therefore, the only way causality can be restored is to add an infinite tower of finely tuned higher spin primaries with $\Delta \sim \Delta'_{\rm gap} \sim \Delta_0$. In other words, addition of a single higher spin operator with $\Delta = \Delta_0$ necessarily brings down the gap to Δ_0.

Let us note that the above argument did not require that this new tower of operators contribute to the TT OPE. For this reason, one might hope that it is possible to fine-tune the higher spin operators such that causality is restored and the gap is still at $\Delta_{\rm gap}$ when considering states created by the stress tensor.

[33]Note that the chaos bound does not directly rule out spin-2 exchange operators. Therefore, one might expect that the causality problem may be resolved by adding a finite number of spin-2 non-conserved single trace primaries. However, it was shown in [5] that non-conserved spin-2 primaries when contribute as exchange operators lead to additional causality violation and hence we will not consider this scenario.

However, this scenario is also not allowed as we explain next. In this case, one can still prove the HNEC starting from the Regge OPE of TT when both operators are smeared appropriately (see [131]). One can then repeat the argument of the previous section to rule out X_ℓ, as well as the entire tower of operators at Δ'_{gap}. Therefore, the only way the tower at $\Delta'_{\text{gap}} \sim \Delta_0$ can lead to a causal CFT is if they also contribute to the TT OPE. In particular, an infinite subset of all higher spin operators must appear in the OPE of the stress tensor (and all low spin operators)

$$TT \sim \sum_J X_J \ . \tag{4.4.1}$$

Let us end this section by summarizing in the gravity language. At the energy scale $E \ll \Lambda$, the dynamics of gravitons is completely determined by the Einstein-Hilbert action. If we wish to add even one higher spin elementary particle ($\ell > 2$) with mass $M_0 \ll \Lambda$, the only way for the theory to remain causal is if we also add an infinite tower of higher spin particles with mass $\sim M_0$. Causality also requires that an infinite subset of these new higher spin particles should be able to decay into two gravitons. As a result, the dynamics of graviton can now be approximated by the the Einstein-Hilbert action only in the energy scale $E \ll M_0$ and hence M_0 is the new cut-off even if we only consider external states created by gravitons.

4.4.2 Stringy Operators above the Gap

We concluded from both gravity and CFT arguments that finitely many higher spin fields with scaling dimensions $\Delta \ll \Delta_{gap}$ are inconsistent with causality even as external operators. We can ask how this result may be modified if we consider external operator X to be a heavy state above the gap, analogous to stringy states in classical string theory.

Let us consider the expectation value of the generalized HNEC operator (4.3.6) in the Hofman-Maldacena states created by a heavy single-trace higher spin operator with spin l. Following [26] we parametrize the leading Regge trajectory as

$$j(\nu) = 2 - \frac{1}{\Delta_{gap}^2}\left(\frac{d^2}{4} + \nu^2\right) + \mathcal{O}\left(\frac{1}{\Delta_{gap}^4}\right) \ . \tag{4.4.2}$$

The external operator has the scaling dimension $\Delta_X \geq \Delta_{gap}$. Consequently, we cannot take the $\Delta_{gap} \to \infty$ limit as before. Instead we must take Δ_{gap} to be large but finite and keep track of terms that may grow in this limit. In the Regge limit $u \to \infty$, with $1 - \rho \gtrsim \frac{\log(u)}{\Delta_{gap}^2}$, we expect the leading trajectory to be nearly flat and integration over the spectral density (4.3.6) to be approximated by the stress-tensor contribution at $\nu = -i\frac{d}{2}$ up to $\frac{1}{\Delta_{\rm gap}^2}$ corrections. This limit is similar to the discussion in section 5.5 in [25] for bounds on real part of phase shift for scattering in AdS. See also discussion about imaginary part of phase shift for AdS scattering in [25, 26, 66].

Therefore, the operator with a positive expectation value is given by[34]

$$u\langle \mathbb{E}_{\Delta(J=2),2}(\rho)\rangle_X = u \sum_{i=0}^{2l} \frac{t^{(i)}}{(1-\rho)^{d-3+i}} + \cdots \ , \tag{4.4.3}$$

where the dots denote terms which are subleading in Δ_{gap}, $t^{(i)}$'s consist of certain combination of OPE coefficients and polarization tensors. The OPE coefficients $t^{(i)}$, are analytic continuation of original OPE coefficients. We have already seen that if the OPE coefficients do not grow with Δ_{gap}, the existence of the operator

[34] The second line follows from the fact that at large Δ_{gap} the saddle point is dominated by the stress-tensor. Here we have assumed that the OPE coefficients do not scale exponentially with increasing Δ_{gap} and hence will not affect the saddle-point.

X is inconsistent with causality. One way in which causality may be restored, is to impose the following gap dependence on the OPE coefficients between heavy operators and the exchange operator[35]:

$$\frac{t^{(i)}}{t^{(0)}} \lesssim \frac{1}{\Delta_{gap}^{i}} \ . \tag{4.4.4}$$

The dependence of OPE coefficients on Δ_{gap} is chosen in (4.4.4) such that higher negative powers of $1-\rho$ would be multiplied by higher powers of $\frac{1}{\Delta_{gap}}$ and consequently become more suppressed in the regime of validity of stress-tensor exchange. This means that we would not get the previous constraints by sending $\rho \to 1$ and as a result, there is no inconsistency with Ward identity or causality for higher spin operators above the gap.

Based on our CFT arguments, (4.4.4) is not fixed to be the unique choice which restores causality. However, this behaviour is very similar to how the scattering amplitude in classical string theory is consistent with causality. The high energy limit of scattering amplitudes in string theory are explored in [152–156]. In addition, generating functions of three point and four point amplitudes for strings on the leading Regge trajectory with arbitrary spin are constructed in [157,158]. Here we focus on a high energy limit of a two to two scattering between closed higher spin strings and tachyons in bosonic string theory. Using the results of [157, 158], the string amplitude is given by the compact expression

$$M(s,t) = (\mathrm{POL})\frac{\Gamma(-\frac{\alpha' s}{4})\Gamma(-\frac{\alpha' t}{4})\Gamma(-\frac{\alpha' u}{4})}{\Gamma(1+\frac{\alpha' s}{4})\Gamma(1+\frac{\alpha' t}{4})\Gamma(1+\frac{\alpha' u}{4})} \ , \tag{4.4.5}$$

where the Mandelstam variables satisfy $s+t+u = \frac{4}{\alpha'}(l-4)$ for closed strings. Here, (POL) represents the tensor structures and polynomials of different momenta. The

[35]In fact, in the case of stress-tensor exchange, Ward identities forces at least one combination of OPE coefficients to grow with $\Delta_X \sim \Delta_{gap}$.

Gamma functions poles in the numerator of (4.4.5) correspond to the exchange of infinitely many higher spin particles with even spins and the mass relation $m(J)^2 = \frac{2}{\alpha'}(J-2)$. In the Regge limit, $s \to \infty$ with t held fixed, the amplitude simplifies to

$$M(s,t) \approx (\text{POL})\frac{\Gamma(-\frac{\alpha' t}{4})}{\Gamma(1+\frac{\alpha' t}{4})}\left(-i\frac{s\alpha'}{4}\right)^{-2+\frac{\alpha' t}{2}} \ . \tag{4.4.6}$$

Note that the Mandelstam variable s plays the same role as u in the CFT analogue. Therefore, to make gravity the dominant force we can either take $\alpha' \to 0$ which corresponds to $\Delta_{\text{gap}} \to \infty$ in the CFT, or take $t \to 0$ which in CFT language is the lightcone limit $\rho \to 0$. In both cases, the polarization part, (POL) becomes

$$\lim_{\alpha' \to 0 \text{ or } t \to 0} (\text{POL}) \propto s^4 \mathcal{E}_{1\mu_1\mu_2\cdots\mu_l}\mathcal{E}_3{}^{\mu_1\mu_2\cdots\mu_l} \ , \tag{4.4.7}$$

where powers of s are dictated by consistency with the gravity result in limits mentioned above. Note that the tensor structure in (4.4.7) is independent of the momenta and does not change sign even if we perform the eikonal experiment in this limit. Thus, in the limit that gravity is dominant, possible causality violating structures are also vanishing and there is no problem with causality. This happens naturally in string theory since there is only one scale α', controlling coefficients in tensor structures, interactions between particles and their masses. As a result, vertices or tensor structures which have higher powers of momentum $\vec{q}$ (analogous to powers of $\frac{1}{1-\rho}$ in CFT) should be accompanied with higher powers of $\sqrt{\alpha'}$ (analogous to powers of $\frac{1}{\Delta_{gap}}$) on dimensional grounds. See also [5, 159] for interesting details of eikonal experiment in string theory.

4.5 Cosmological Implications

The bound on higher spin particles has a natural application in inflation. The epoch of inflation is a quasi de Sitter expansion of the universe, immediately after the big bang. The primordial cosmological fluctuations produced during inflation naturally explains the observed temperature fluctuations of cosmic microwave background (CMB) and the large-scale structures of the universe. If higher spin particles were present during inflation, they would affect the behavior of primordial cosmological fluctuations. In particular, higher spin particles would produce distinct signatures on the three-point function of scalar perturbations in the squeezed limit. Hence, the bound on higher spin particles imposes rather strong constraints on these three-point functions.

Consider one or more higher spin particles during inflation. The approximate de Sitter symmetry during inflation dictates that mass of any such particle, even before we impose our causality constraints, must satisfy the Higuchi bound [160, 161]

$$m^2 > \ell(\ell-1)H^2 \ , \tag{4.5.1}$$

where, H is the Hubble rate during inflation. Particles with masses that violate the Higuchi bound correspond to non-unitary representations in de Sitter space, so the Higuchi bound is analogous to the unitarity bound in CFT.[36] The bound on higher spin particles obtained in this chapter are valid in flat and AdS spacetime. We will not attempt to derive similar bounds directly in de Sitter. Instead, we will adopt the point of view of [5, 65] and assume that the same bounds hold even

[36]We should note that certain discrete values of mass below the Higuchi bound are also allowed. See [162] for a nice review.

in de Sitter spacetime. This is indeed a reasonable assumption since these bounds were obtained by studying local high energy scattering which is insensitive to the spacetime curvature. Therefore, in de Sitter spacetime in Einstein gravity, any additional elementary particle with spin $\ell > 2$ cannot have a mass $m \lesssim \Lambda$, where Λ is the scale of new physics in the original effective action. In any sensible low energy theory we must have $H \ll \Lambda$ and hence the causality bound is stronger than the Higuchi bound. Furthermore, the causality bound also implies that all elementary higher spin particles must belong to the principal series of unitary representation of the de Sitter isometry group.

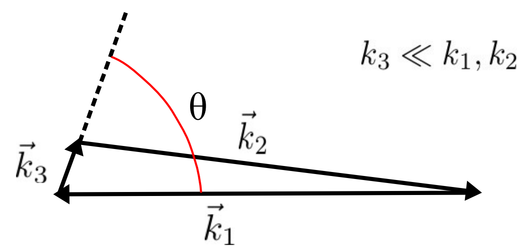


Figure 4.7: The squeezed limit of three-point functions.

Inflation naturally predicts that the scalar curvature perturbation ζ produced during inflation is nearly scale invariant and Gaussian. The momentum space three-point function of the scalar curvature perturbation $\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle$ is a good measure of the deviation from exact Gaussianity. Higher spin particles affect the three-point function of scalar perturbations in a unique way. In an inflating universe, the massive higher spin particles can be spontaneously created. It was shown in [135] that the spontaneous creation of higher spin particles produces characteristic signatures on the late time three-point function of scalar fluctuations. In particular, in the squeezed limit $k_1, k_2 \gg k_3$ (see figure 4.7), the late time scalar three-point function admits an expansion in spin of the new particles present during inflation:[37]

$$\frac{\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle}{\langle\zeta(\vec{k}_1)\zeta(-\vec{k}_1)\rangle\langle\zeta(\vec{k}_3)\zeta(-\vec{k}_3)\rangle} \sim \epsilon M_{Pl}^2 \sum_\ell \lambda_\ell^2 \; I_\ell\left(\frac{m_\ell}{H}, \frac{k_3}{k_1}\right) P_\ell(\cos\theta) \;, \qquad (4.5.2)$$

where ϵ is one of the slow roll parameters and λ_ℓ is the coupling between ζ and the higher spin particle with mass m_ℓ and spin ℓ. $P_\ell(\cos\theta)$ is the Legendre polynomial whose index is fixed by the spin of the particle and θ is the angle between vectors $\vec{k}_1$ and $\vec{k}_3$. The exact form of the function $I_\ell\left(\frac{m_\ell}{H}, \frac{k_3}{k_1}\right)$ can be found in [135]. The bound on higher spin particles from causality implies that $m_\ell \sim \Lambda \gg H$ for $\ell > 2$ and hence

$$I_\ell\left(\frac{m_\ell}{H}, \frac{k_3}{k_1}\right) \sim -\pi^2 e^{-\frac{2\pi\Lambda}{H}} \left(\frac{\Lambda}{H}\right)^{2\ell-3} \left(\frac{k_3}{k_1}\right)^{3/2} \mathrm{Re}\left[e^{\frac{i\pi}{4}}\left(\frac{k_3}{4k_1}\right)^{i\frac{\Lambda}{H}}\right] . \qquad (4.5.3)$$

The oscillatory behavior of the above expression is a consequence of a quantum interference effect between two different processes [135]. Moreover, the above expression also implies that contributions of higher spins to the three-point function in the squeezed limit must be exponentially suppressed. The exponential suppression can be understood as the probability for the spontaneous production of massive higher spin particles in the principal series at de Sitter temperature $T_{dS} = H/2\pi$.

Now, if I_ℓ with $\ell > 2$ is detected in future experiments, then the scale of new physics must be $\Lambda \sim H$. This necessarily requires the presence of not one but an infinite tower of higher spin particles with spins $\ell > 2$ and masses comparable to the Hubble scale. This scenario is very similar to string theory. Any detection of I_ℓ with $\ell > 2$ can be interpreted as evidence in favor of string theory with the string scale comparable to the Hubble scale and a very weak coupling which explains small H/M_{pl}.

[37]For simplicity of notation, we are omitting the Dirac delta functions.

It is obvious from (4.5.2) that the effects of higher spin particles are always suppressed by the slow roll parameter and hence not observable in the near future. The derivation of (4.5.2) relied heavily on the approximate conformal invariance of the inflationary background. This approximate conformal invariance is also responsible for the slow roll suppression. However, if we allow for a large breaking of conformal invariance, the signatures of massive higher spin particles can be large enough to be detected by future experiments. In particular, using the framework of effective field theory of inflation it was shown in [163] that there are interesting scenarios in which higher spin particles contribute significantly to the scalar non-Gaussanity. Furthermore, it was shown in [163] that higher spin particles can also produce detectable as well as distinctive signatures on the scalar-scalar-graviton three-point function in the squeezed limit. Experimental exploration of this form of non-Gaussanity through the measurement of the $\langle BTT \rangle$ correlator of CMB anisotropies can actually be a reality in the near future [163]. In fact, in the most optimistic scenario, the proposed CMB Stage IV experiments [164] will be sensitive enough to detect massive higher spin particles, providing indirect evidence in favor of a theory which is very similar to low scale string theory.

APPENDIX A

APPENDICES

A.1 Witten diagram calculations

In this appendix, we derive some of the equations in section 2.3 by evaluating Witten diagrams in the Regge limit.

A.1.1 Feynman rules

Scalar bulk-to-boundary propagator

The scalar bulk-to-boundary propagator between a bulk point (z, x) and a boundary point x' in Euclidean AdS_{d+1} is given by

$$D(z, x; x') = c_\Delta \frac{z^\Delta}{(z^2 + |\vec{x} - \vec{x}'|^2)^\Delta} \ , \tag{A.1.1}$$

where

$$c_\Delta = \frac{\Gamma(\Delta)}{\pi^{d/2}\Gamma(\Delta - d/2)} \ . \tag{A.1.2}$$

$\Delta = \frac{1}{2}(d + \sqrt{d + 4m^2})$ is the scaling dimension of the boundary operator which is dual to a bulk scalar field of mass m.

Graviton bulk-to-bulk propagator

Let us now write down the graviton bulk-to-bulk propagator between point (z_1, y_1) and (z_2, y_2) in Euclidean AdS$_{d+1}$. We first introduce the quantity

$$\mathcal{U} = -1 + \frac{1}{2z_1z_2}\left(z_1^2 + z_2^2 + (y_1 - y_2)^2\right) \ , \qquad \zeta = \frac{2z_1z_2}{z_1^2 + z_2^2 + (y_1 - y_2)^2} \ . \quad \text{(A.1.3)}$$

Let us also introduce

$$G(\mathcal{U}) = (8\pi G_N)\frac{\Gamma(d)\Gamma\left(\frac{d+1}{2}\right)\zeta^d}{2\pi^{(d+1)/2}\Gamma(d+1)}{}_2F_1\left(\frac{d}{2}, \frac{1+d}{2}, \frac{d}{2} + 1, \zeta^2\right) \ . \qquad \text{(A.1.4)}$$

The bulk-to-bulk propagator is given by [?]

$$G_{\mu\nu\mu'\nu'} = \left(\partial_\mu\partial_{\mu'}\mathcal{U}\partial_\nu\partial_{\nu'}\mathcal{U} + \partial_\mu\partial_{\nu'}\mathcal{U}\partial_\nu\partial_{\mu'}\mathcal{U}\right)G(\mathcal{U}) + g_{\mu\nu}g_{\mu'\nu'}H(\mathcal{U}) \qquad \text{(A.1.5)}$$

where,

$$H(\mathcal{U}) = -\frac{2(1+\mathcal{U})^2}{(d-1)}G(\mathcal{U}) + \frac{2(d-2)(1+\mathcal{U})}{(d-1)}\int_{\mathcal{U}}^\infty du'G(u') \ . \qquad \text{(A.1.6)}$$

Graviton-scalar-scalar vertex

The graviton-scalar-scalar vertex is given by the bulk stress tensor of the scalar field in terms of the scalar bulk-to-boundary propagators

$$T_{\mu\nu}^{bulk}(D_1^\psi; D_2^\psi) = -\mathcal{N}\left(\partial_\mu D_1^\psi\partial_\nu D_2^\psi + \partial_\nu D_1^\psi\partial_\mu D_2^\psi - (\partial_\alpha D_1^\psi)(\partial^\alpha D_2^\psi)g_{\mu\nu}^{AdS} - m^2 D_1^\psi D_2^\psi g_{\mu\nu}^{AdS}\right) \ . \qquad \text{(A.1.7)}$$

where,

$$D_1^\psi \equiv D^\psi(z, x; x_1) \ , \qquad D_2^\psi \equiv D^\psi(z, x; x_2) \qquad \text{(A.1.8)}$$

and

$$\mathcal{N} = \frac{1}{2c_{\Delta_\psi}(2\Delta_\psi - d)} \ . \qquad \text{(A.1.9)}$$

Note that all the derivatives in equation (A.1.7) are taken with respect to the bulk point (z, x).

A.1.2 Vertex diagram

Now consider the scalar-scalar-graviton vertex diagram, shown in (2.3.8). We set $D = 4$, but the result easily generalizes. ψ is a heavy operator with $c_T \gg \Delta_\psi \gg 1$. In general, this diagram is given by

$$\Pi_{\alpha'\beta'}(x_1, x_2; z', x') = i \int d^d x dz \sqrt{-g^{AdS}} G^{\mu\nu}_{\alpha'\beta'}(z, x; z', x') T^{bulk}_{\mu\nu}(D^\psi(z, x; x_1); D^\psi(z, x; x_2)) \,, \tag{A.1.10}$$

where, $G^{\mu\nu}_{\alpha'\beta'}(z, x; z', x')$ is the graviton bulk-to-bulk propagator and $D^\psi(z, x; x')$ is the scalar bulk-to-boundary propagator. $T^{bulk}_{\mu\nu}$ is the bulk stress tensor of the dual scalar field in terms of the scalar bulk-to-boundary propagator (A.1.7). $\Pi_{\alpha'\beta'}(x_1, x_2; z', x')$ can easily be computed by using the results of [165] (and also [166]). For, $D = 4$, following [165], we can write down

$$\Pi_{\alpha'\beta'}(x_1, x_2; z', x') = -\frac{\Delta_\psi}{4\pi^2} \frac{8\pi G_N}{x_{12}^{2\Delta_\psi}} \frac{1}{z'^2} \left(\frac{1}{3}\eta_{\alpha'\beta'} - J_{\alpha' z'}(X' - X_1) J_{\beta' z'}(X' - X_1) \right) f(t) + \cdots \,, \tag{A.1.11}$$

where $X_1 = (z = 0, x_1)$, $X' = (z, x')$ and dots represent gauge dependent terms which will not contribute to the final answer. The inversion tensor $J_{\alpha\beta}(X' - X_1)$ is given by

$$J_{\alpha\beta}(X' - X_1) = \eta_{\alpha\beta} - \frac{2(X' - X_1)_\alpha (X' - X_1)_\beta}{z^2 + |x' - x_1|^2} \,. \tag{A.1.12}$$

with indices raised and lowered by $\eta_{\alpha\beta}$. The function $f(t)$ is given by [165, 166]

$$f(t) = \frac{t(1 - t^{\Delta_\psi - 1})}{(1 - t)} \,, \qquad t = \frac{z'^2 x_{12}^2}{(z'^2 + (x' - x_1)^2)(z'^2 + (x' - x_2)^2)} \,. \tag{A.1.13}$$

Note that equation (A.1.11) is not symmetric with respect to x_1 and x_2. This asymmetry, as noted in [166], is a consequence of dropping the gauge dependent terms. In an actual correlator, there will be an integral over (z', x') which will make the final answer symmetric under $x_1 \leftrightarrow x_2$.

So far we have not assume anything about Δ_ψ. Let us now take the limit $\Delta_\psi \to \infty$. In this limit we can approximate

$$f(t) \approx \frac{t}{1-t} \ . \tag{A.1.14}$$

In the Euclidean signature, $t \leq 1$ and hence the term t^{Δ_ψ} can be ignored.[1]

Let us now choose $x_1 = (u, v, \vec{0})$ and $x_2 = -x_1$ with $u > 0, v < 0$. In the Regge limit (2.3.1), using (A.1.11) along with (A.1.14), we obtain

$$\Pi_{\alpha'\beta'}(x_1, x_2; z', x') = i\frac{(8\pi G_N)\Delta_\psi}{16\pi}\frac{1}{(-4uv)^{\Delta_\psi}}\frac{u}{z'\sqrt{-uv}}\frac{(r-1)^3}{r(r+1)}\delta^v_{\alpha'}\delta^v_{\beta'}\delta(v') + \mathcal{O}\left(u^0, \frac{1}{\Delta_\psi}\right) \ , \tag{A.1.15}$$

where (recall that $x' = (u', v', \vec{x}')$),

$$r = \sqrt{\frac{\vec{x}'^2 + \left(z' - \sqrt{-uv}\right)^2}{\vec{x}'^2 + \left(z' + \sqrt{-uv}\right)^2}} \ . \tag{A.1.16}$$

This is exactly the planar shockwave solution (2.2.1) in AdS_5, with an additional factor of i, once we identify $z_0 = \sqrt{-uv}$. Thus we have derived (2.3.13).

To rewrite this as the null geodesic integral (2.3.9), note that the bulk-to-bulk graviton propagator $G_{uu\alpha'\beta'}(u, v = 0, \vec{x} = 0, z = z_0; z', x')$ can be written as $P_{uu\alpha'\beta'}G(u = 0, v, \vec{x} = 0, z = z_0; z', x')$, where $P_{uu\alpha'\beta'}$ is independent of u and $G(u = 0, v, \vec{x} = 0, z = z_0; z', x')$ is the scalar bulk-to-bulk propagator. Therefore,

$$\begin{aligned}\int du G_{uu\alpha'\beta'}(u, v = 0, \vec{x} = 0, z = z_0; z', x') &= P_{uu\alpha'\beta'}\int du G(u, v = 0, \vec{x} = 0, z = z_0; z', x') \\ &\equiv P_{uu\alpha'\beta'}\, g(z', v', \vec{x}') \ ,\end{aligned} \tag{A.1.17}$$

[1] Here we compute the Euclidean answer, then analytically continue. This is valid as long as we do not go so far into the Regge regime as to compete with the large-Δ_ψ limit. The same final formula can be obtained by taking u, v imaginary, as in the shockwave state discussed in section 2.2, and evaluating the diagram by a saddlepoint approximation.

where, in $(d+1)-$dimensions $g(z',v',\vec{x}')$ satisfies the differential equation

$$\left(\partial_{z'}^2 - (d-1)\frac{\partial_{z'}}{z'} + \partial_{\vec{x}'^2}\right) g(z',v',\vec{x}') = -2i\,(z_0)^{d-1}\,\delta(v')\delta(\vec{x}')\delta(z'-z_0)\ . \quad \text{(A.1.18)}$$

$g(z',v',\vec{x}')$ satisfies exactly the same differential equation as a planar shockwave in AdS_{d+1} (see for example [5]) along with the same boundary condition and hence

$$g(z',v',\vec{x}') = 2i\delta(v')\frac{z'z_0(4\pi)^{\frac{1-d}{2}}\Gamma(\frac{d+1}{2})}{d(d-1)}\left(\frac{\rho^2}{1-\rho^2}\right)^{1-d} {}_2F_1\left(d-1,\frac{d+1}{2},d+1,-\frac{1-\rho^2}{\rho^2}\right)\ , \quad \text{(A.1.19)}$$

where,

$$\rho = \sqrt{\frac{(z'-z_0)^2+\vec{x}'^2}{(z'+z_0)^2+\vec{x}'^2}}\ . \quad \text{(A.1.20)}$$

Therefore, with $D=4$,

$$\int du G_{uu\alpha'\beta'}(u,v=0,\vec{x}=0,z=z_0;z',x') = -\frac{iG_N}{z'z_0}\frac{(\rho-1)^3}{\rho(\rho+1)}\delta^v_{\alpha'}\delta^v_{\beta'}\delta(v') \quad \text{(A.1.21)}$$

which allows us to rewrite (A.1.15) as

$$\Pi_{\alpha'\beta'}(x_1,x_2;z',x') = -\frac{\Delta_\psi u}{2}\int_{-\infty}^{\infty} du'' G_{uu\alpha'\beta'}(u'',v=0,\vec{x}=0,z=\sqrt{-uv};z',x')\ . \quad \text{(A.1.22)}$$

This is (2.3.9) in the main text.

A.1.3 4-point functions

Now consider the four-point function $\langle\psi(u,v)\mathcal{O}(x_3)\mathcal{O}(x_4)\psi(-u,-v)\rangle$ in the Regge limit, where $\mathcal{O}$ is an arbitrary operator with or without spin. This four-point function can be computed from the bulk side using Witten diagrams, involving the vertex function computed above. The exchange diagram gives

$$\langle\psi(u,v)\mathcal{O}(x_3)\mathcal{O}(x_4)\psi(-u,-v)\rangle = \langle\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle\langle\psi(x_1)\psi(x_2)\rangle \quad \text{(A.1.23)}$$
$$+ 2i\int d^4x'dz'\sqrt{-g^{AdS}}T_{\mathcal{O}}^{\alpha'\beta'}(z,x';x_3,x_4)\Pi_{\alpha'\beta'} + \cdots\ ,$$

where $T_{\mathcal{O}}^{\alpha'\beta'}$ is the bulk stress tensor of the field dual to the operator $\mathcal{O}$. Using (A.1.15), this implies

$$\frac{\langle\psi(u,v)\mathcal{O}(x_3)\mathcal{O}(x_4)\psi(-u,-v)\rangle_{\rm Regge}}{\langle\psi(x_1)\psi(x_2)\rangle} = \langle\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle_{shock} \ , \tag{A.1.24}$$

where $\langle\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle_{shock}$ is the two-point function computed in the imaginary shockwave (2.3.11).

A.2 Check of the stress-tensor block

In this appendix, we use the Regge OPE (2.3.15) to reproduce the scalar conformal block for stress tensor exchange in the Regge limit in $D=4$. Let us consider the correlator

$$\langle\psi(u,v)\phi(x=-1)\phi(x=1)\psi(-u,-v)\rangle \ . \tag{A.2.1}$$

In the Regge limit (2.3.1), the contribution from the stress tensor according to (2.3.15) is

$$\begin{aligned} g_T(z,\bar{z}) = \frac{2\lambda_T}{\pi^2 v}\int_{t'^2+x'^2+\vec{x}'^2<-uv} dt'dx'd^2\vec{x}' \\ \times\int_{-\infty}^{\infty} du'\langle\phi(x=-1)T_{uu}\left(\frac{u'}{2}+t',-\frac{u'}{2}+ix',i\vec{x}'\right)\phi(x=1)\rangle \ , \end{aligned} \tag{A.2.2}$$

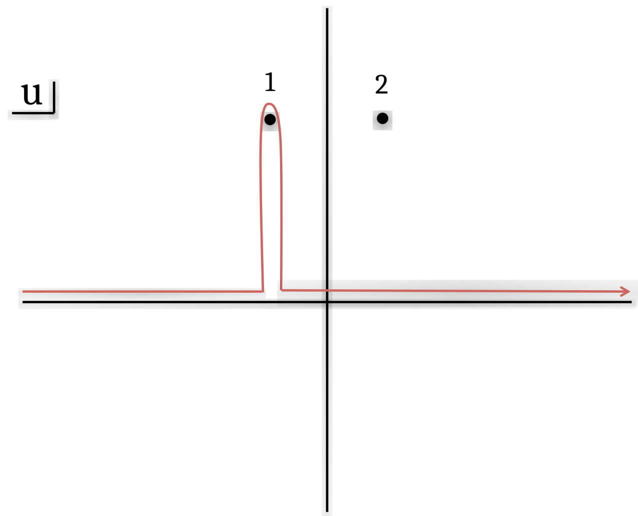
where in the Regge limit $z=4/u$, $\bar{z}=-4v$. The u-integral is done first and then the remaining integral. The u-integral is subtle, as discussed in section 2.3.1: It is defined in such a way that the u-contour circles one pole. First, let us write

$$u=-\eta\sigma \ , \qquad v=\frac{1}{\sigma} \tag{A.2.3}$$

where, the Regge limit is obtained by taking $\sigma\to 0$. Note that the conformal cross-ratios in the Regge limit are

$$\bar{z}=4\eta\sigma \ , \qquad z=4\sigma \ . \tag{A.2.4}$$

We now perform the u-integral. If we do the u-integral along the real line, then the u-contour does not enclose any poles and hence the integral vanishes. Instead, to obtain the ordering (A.2.1), the u-integral must be done along the following contour:



(A.2.5)

where the poles 1 and 2 are due to the operators $\phi(x = 1)$ and $\phi(x = -1)$ respectively. This means that the u integral can be evaluated by a residue, with the result

$$g_T = \frac{10\Delta_\psi\Delta_\phi}{\pi^6 c_T v}\int_{t'^2+x'^2+\vec{x}'^2<-uv} dt'dx'd^2\vec{x}'\frac{(t'+ix'-1)^2(t'+ix'+1)^2}{(t'^2+2it'x'-x'^2+\vec{x}'^2-1)^3}\ . \tag{A.2.6}$$

Now, one can perform the $\vec{x}'$-integrals and then t' and x' integrals, yielding

$$g_T(z,\bar{z}) = i\left(\frac{40\Delta_\psi\Delta_{\mathcal{O}}}{c_T\pi^3}\right)\frac{\bar{z}}{z(z-\bar{z})}\ . \tag{A.2.7}$$

Note that we could have also used (2.3.16) because of the way the contours are defined.

Let's compare (A.2.7) to the known conformal block, as computed by Dolan and Osborn [151]. In 4d, the conformal block for stress tensor exchange, including the OPE coefficient set by the Ward identity, is

$$g_T^{full}(z,\bar{z}) = -\left(\frac{4\Delta_\psi\Delta_O}{9c_T\pi^4}\right)\frac{z^4\bar{z}\,{}_2F_1(3,3,6,z) - \bar{z}^4 z\,{}_2F_1(3,3,6,\bar{z})}{4(z-\bar{z})} \tag{A.2.8}$$

The hypergeometric function is ${}_2F_1(3,3,6,z) = \frac{90}{z^4}(z-2) - \frac{30}{z^5}(z^2-6z+6)\log(1-z)$. To reach the Regge limit, we first take $\log(1-z) \to \log(1-z) + 2\pi i$, then $z, \bar{z} \to 0$. The leading term is exactly (A.2.7).

A.2.1 Time delay from the Dolan-Osborn block

In section 2.4.2 we derived the gravitational time delay Δv using the Regge OPE. Here, for comparison, we repeat the CFT calculation in $D = 4$ using the full Dolan-Osborn conformal block, quoted in (A.2.8). The cross-ratios in the kinematics (2.4.3), (2.4.5), with $\vec{x}_\phi = 0$ and $\delta \ll 1$, are

$$z = -\frac{4i\delta}{u_\phi}, \qquad \bar{z} = \frac{4iv_\phi\delta}{z_0^2}\ . \tag{A.2.9}$$

Plugging these values into (A.2.7) gives exactly the gravity result without winding, on the first line of (2.4.11). This is the case where we reach the Regge regime by sending z around 1. If we instead send $\bar{z}$ around 1, or equivalently exchange $z \leftrightarrow \bar{z}$ in (A.2.9) and apply (A.2.7), then the result matches the second line of (2.4.11).

A.3 Derivation of the general Lightcone and Regge block

In this appendix, we find a general formula for the OPE block of (scalar)×(scalar) → (anything) in the Regge limit, assuming x_1 and x_2 are timelike separated. The result is analytically continued to spacelike separation in (3.9.1). Using the shadow operator formalism, the OPE block is written as an integral over a causal diamond, in the future of x_1 and past of x_2. In lightcone or Regge limit, the diamond will shrink to a line or a slab respectively, and the OPE formula simplifies.

Following [39,40], we write the OPE as

$$\frac{\psi(x_1)\psi(x_1)}{\langle\psi(x_2)\psi(x_1)\rangle} = \mathcal{N}_X \int_{D(x_1,x_2)} d^d\zeta \left(\frac{(x_1-\zeta)^2(x_2-\zeta)^2}{x_{21}^2}\right)^{\frac{\Delta-d}{2}} \hat{t}_{12}^{\mu_1}\hat{t}_{12}^{\mu_2}\cdots\hat{t}_{12}^{\mu_l}\, X_{\mu_1\mu_2\cdots\mu_\ell} \tag{A.3.1}$$

where $\mathcal{N}_X$ is a normalization constant to be determined below, and

$$t_{12}^{\mu} = \hat{t}_{12}^{\mu}|t_{12}| = \frac{(x_2-\zeta)^\mu}{(x_2-\zeta)^2} - \frac{(x_1-\zeta)^\mu}{(x_1-\zeta)^2}\ . \tag{A.3.2}$$

The kinematics are the same as the main text,

$$x_2 = (u, v, \vec{0})\ , \qquad x_1 = (-u, -v, \vec{0})\ , \qquad \zeta = (\tilde{u}, \tilde{v}, \vec{x})\ . \tag{A.3.3}$$

However, here we assume both u and v are positive so the diamond $D(x_1,x_2)$ is real. The diamond is the region in spacetime defined by

$$\frac{\vec{x}^2}{\tilde{u}+u} - v \le \tilde{v} \le v - \frac{\vec{x}^2}{u-\tilde{u}}\ , \qquad \vec{x}^2 \le uv\left(1-\frac{\tilde{u}^2}{u^2}\right)\ , \qquad -u \le \tilde{u} \le u\ . \tag{A.3.4}$$

In both the Regge and lightcone limits, $v \to 0$, so $\tilde{v} \to 0$ as well. Hence, we can set $\tilde{v} = 0$ in the argument of X and integrate the expression over v. For the same reason, we keep only $\mu = u$ in t_{12}^{μ}. In either limit, integration over $\tilde{v}$ yields:

$$\frac{\psi(x_2)\psi(x_1)|_X}{\langle\psi(x_2)\psi(x_1)\rangle} \sim \frac{u^{\ell-1}}{(uv)^{\frac{\Delta+\ell-d}{2}}} \int du d^{d-2}\vec{x} \frac{\left[uv\left(1-\frac{\tilde{u}^2}{u^2}\right) - \vec{x}^2\right]^{\Delta-d+1}}{\left(1-\frac{\tilde{u}^2}{u^2}\right)^{\frac{\Delta-d-\ell}{2}+1}} X_{uu\cdots u}(u, 0, \vec{x})\ . \tag{A.3.5}$$

In the lightcone limit, $uv \to 0$, one can also integrate over the transverse part in A.3.5 and we recover the OPE as a line integral appearing in [10]:

$$\frac{\psi(x_2)\psi(x_1)|_X^{L.C}}{\langle\psi(x_2)\psi(x_1)\rangle} \sim u^{\ell-1}(uv)^{\frac{\Delta-\ell}{2}} \int_{-u}^{u} d\tilde{u}(1-\tilde{u}^2/u^2)^{\frac{\Delta+\ell}{2}-1} X_{uu\cdots u}(\tilde{u}, 0, \vec{0})\ . \tag{A.3.6}$$

Taking the Regge limit of (A.3.5), $v \to 0, uv =$ fixed, we find:

$$\frac{\psi(x_2)\psi(x_1)|_X^{\text{Regge}}}{\langle\psi(x_2)\psi(x_1)\rangle} \sim \frac{(uv)^{\frac{d-\Delta-\ell}{2}}}{u^{1-\ell}} \int_{-\infty}^{+\infty} d\tilde{u} \int_{\vec{x}^2 \leq uv} d^{d-2}\vec{x}\,\left(uv - \vec{x}^2\right)^{\Delta-d+1} X_{uu\cdots u}(\tilde{u}, 0, \vec{x}) \ . \tag{A.3.7}$$

Finally by taking $uv \to 0$ after the Regge limit and comparing to the lightcone result in [10], we find the constant of proportionality. The final answer including factors is:

$$\begin{aligned} \frac{\psi(x_2)\psi(x_1)|_X^{\text{Regge}}}{\langle\psi(x_2)\psi(x_1)\rangle} &= (-1)^{\frac{\Delta-l}{2}}\pi^{\frac{1-d}{2}}2^{\Delta}\frac{\Gamma(\frac{\Delta+\ell+1}{2})}{\Gamma(\frac{\Delta+\ell}{2})}\frac{\Gamma(\Delta-d/2+1)}{\Gamma(\Delta-d+2)} \\ &\frac{C_{\psi\psi X}}{C_X}\frac{(uv)^{\frac{d-\Delta-\ell}{2}}}{u^{1-\ell}}\int_{-\infty}^{+\infty} d\tilde{u}\int_{\vec{x}^2\leq uv} d^{d-2}\vec{x}(uv-\vec{x}^2)^{\Delta-d+1}X_{uu\cdots u}(\tilde{u},0,\vec{x}) \ , \end{aligned} \tag{A.3.8}$$

where $C_{\psi\psi X}$ is the OPE coefficient and C_X is the normalization of the two-point function $\langle XX\rangle$.

A.4 Smearing the Regge amplitude

In this appendix, we give a simple example that illustrates how smearing projects out double-trace operators in the Regge amplitude. This gives another perspective on the results of section 2.5.4.

Consider the correlator $\langle\phi\phi\psi\psi\rangle$, where ϕ and ψ are scalar primaries with $\Delta_\phi = 2$ and $\Delta_\psi \gg 1$. The Regge amplitude corresponding to graviton exchange in the bulk, obtained by doing the ν integral in (2.5.27), is

$$G - 1 \sim \frac{z\bar{z}}{(z+\bar{z})^3} \ . \tag{A.4.1}$$

(We will not keep track of the overall factor). Note that, unlike the stress tensor conformal block, this is regular at $z = \bar{z}$; the apparent singularity from stress tensor exchange is smoothed out by the double trace contributions. Nonetheless we will find singular behavior at $z_{center} = \bar{z}_{center}$ after smearing.

The smearing integral, with the kinematics from section 2.5.4, is then

$$\delta G \sim \int_0^\infty r dr \int_{-\infty}^{\infty} dy \frac{1}{\left(\frac{(t_0+iy_1)(t_0-iy_2)}{z_0^2} x_{12}^2\right)^{\Delta_\phi}} \frac{z\bar{z}}{(z+\bar{z})^3} \tag{A.4.2}$$

$$= \int_0^\infty r dr \int_{-\infty}^{\infty} dy \frac{z_0^2}{2\left(r^2+4t_0^2+y^2\right)\left(2t_0\left(z_0^2+1\right)-iy\left(z_0^2-1\right)\right)^3} \tag{A.4.3}$$

$$= \frac{\pi}{16 t_0} \frac{z_{center}}{\bar{z}_{center}(z_{center}-\bar{z}_{center})} \,. \tag{A.4.4}$$

Smearing just the stress tensor block gives the same result, so the double-trace operators have indeed been projected out.

A.5 Regge amplitude for $\langle JJ\psi\psi\rangle$

In this appendix we give some details of the calculation in section 2.6.2. The differential operators $\hat{D}^{(1)\mu,\nu}$ are given explicitly by

$$\begin{aligned}
\epsilon_\mu\epsilon_\nu \hat{D}^{(1)\mu,\nu} =& u^2 V_{124}V_{213}\partial_u^2 - v^2(V_{123}-V_{124})(V_{213}-V_{214})\partial_v^2 \\
&+ u\left(V_{124}V_{213} - \frac{1}{2}H_{12}\right)\partial_u - v(V_{123}-V_{124})(V_{213}-V_{214})\partial_v \\
&- uv(V_{123}V_{213} + V_{124}(V_{214}-2V_{213}))\partial_u\partial_v \,, \\
\epsilon_\mu\epsilon_\nu \hat{D}^{(2)\mu,\nu} =& H_{12} \,, \\
\epsilon_\mu\epsilon_\nu \hat{D}^{(3)\mu,\nu} =& u^2 v V_{123}V_{214}\partial_u^2 + u\left(vV_{123}V_{214} - \frac{1}{2}H_{12}\right)\partial_u \\
&- uv(V_{123}(V_{213}-2vV_{214}) + V_{124}V_{214})\partial_u\partial_v \\
&+ (vV_{123}-V_{124})(vV_{214}-V_{213})\partial_v \\
&+ v(vV_{123}-V_{124})(vV_{214}-V_{213})\partial_v^2 \,,
\end{aligned} \tag{A.5.1}$$

with $u = z\bar{z}, v = (1-z)(1-\bar{z})$ and conformal structures defined as

$$
\begin{aligned}
H_{ij} &= -2\left(-(x_{ij}\cdot\epsilon_i)(x_{ji}\cdot\epsilon_j) - \frac{1}{2}(\epsilon_i\cdot\epsilon_j)(x_{ij}\cdot x_{ij})\right)\ , \\
V_{ijk} &= -\frac{2\left(\frac{1}{2}(x_{ij}\cdot x_{ij})(x_{ki}\cdot\epsilon_i) - \frac{1}{2}(x_{ik}\cdot x_{ik})(x_{ji}\cdot\epsilon_i)\right)}{x_{jk}\cdot x_{jk}}\ .
\end{aligned}
\tag{A.5.2}
$$

The coefficients $a^{(k)}(\nu)$ introduced in 2.6.2 are given by

$$
\begin{aligned}
a^{(1)}(\nu) =& \frac{10\pi\nu n_f\Gamma\left(4-\frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+4\right)\Gamma\left(\Delta_\psi-\frac{i\nu}{2}\right)\Gamma\left(\Delta_\psi+\frac{i\nu}{2}\right)}{\left(\nu^2+4\right)\Gamma(\Delta_\psi-1)\Gamma(\Delta_\psi+1)}\ , \\
a^{(2)}(\nu) =& \frac{5\pi\nu\Gamma\left(4-\frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+4\right)(12n_f+n_s)\Gamma\left(\Delta_\psi-\frac{i\nu}{2}\right)\Gamma\left(\Delta_\psi+\frac{i\nu}{2}\right)}{2\left(\nu^2+4\right)\Gamma(\Delta_\psi-1)\Gamma(\Delta_\psi+1)}\ , \\
a^{(3)}(\nu) =& -\frac{5\pi\nu\Gamma\left(4-\frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+4\right)(4n_f+n_s)\Gamma\left(\Delta_\psi-\frac{i\nu}{2}\right)\Gamma\left(\Delta_\psi+\frac{i\nu}{2}\right)}{2\left(\nu^2+4\right)\Gamma(\Delta_\psi-1)\Gamma(\Delta_\psi+1)}\ .
\end{aligned}
\tag{A.5.3}
$$

The full Regge amplitude in (2.6.4) is

$$
\begin{aligned}
&\epsilon_\mu\tilde{\epsilon}_\nu\delta G^{\mu,\nu} = -2\int d\nu\frac{5\pi\nu\Gamma\left(4-\frac{i\nu}{2}\right)\Gamma\left(\frac{i\nu}{2}+4\right)z^{-\frac{i\nu}{2}}\bar{z}^{\frac{i\nu}{2}}\Gamma\left(\Delta_\psi-\frac{i\nu}{2}\right)\Gamma\left(\Delta_\psi+\frac{i\nu}{2}\right)}{8\left(\nu^2+4\right)\Gamma(\Delta_\psi-1)\Gamma(\Delta_\psi+1)(z-\bar{z})^5} \\
&4n_f(12H_{12}z^2+\nu^2V_{124}V_{213}z^2-4i\nu V_{124}V_{213}z^2-4V_{124}V_{213}z^2-24\bar{z}H_{12}z-2\bar{z}\nu^2V_{124}V_{213}z-8\bar{z}V_{124}V_{213}z+8i\nu V_{124}V_{213}z \\
&+8V_{124}V_{213}z+12\bar{z}^2H_{12}-4\bar{z}^2V_{124}V_{213}+\bar{z}^2\nu^2V_{124}V_{213}+8\bar{z}V_{124}V_{213}+4i\bar{z}^2\nu V_{124}V_{213}-8i\bar{z}\nu V_{124}V_{213} \\
&-(z-1)(\bar{z}-1)((z-\bar{z})^2\nu^2-4i(z-\bar{z})(z+\bar{z}-2)\nu-4(z+\bar{z}-2)(z+\bar{z}))V_{123}V_{214})(z-\bar{z})^2+n_s(((-i\nu+\bar{z}(2i\nu+2)+4)H_{12} \\
&+(\bar{z}-1)(V_{123}((4-2\nu(-3i+\nu))V_{213}+(5\nu^2+4i(\bar{z}-5)\nu+8(\bar{z}-2))V_{214})-2(\nu(-3i+\nu)-2)V_{124}V_{214}))z^4+2((\nu^2+12)V_{123}V_{214}\bar{z}^3 \\
&+((-3i\nu-1)H_{12}+(\nu(3i+\nu)+10)V_{123}V_{213}+((\nu(3i+\nu)+10)V_{124}-(\nu(2i+3\nu)+34)V_{123})V_{214})\bar{z}^2+(i(10i+\nu)H_{12} \\
&+2(-2i+\nu)((-4i+\nu)V_{123}-(-i+\nu)V_{124})V_{213}-2(-4i+\nu)((-4i+\nu)V_{123}-(-2i+\nu)V_{124})V_{214})\bar{z}+(-i+\nu)(V_{123}(2(-5i+2\nu)V_{214} \\
&-3(-2i+\nu)V_{213})+V_{124}(2(-i+\nu)V_{213}-3(-2i+\nu)V_{214})))z^3+2(2(2-i\nu)V_{123}V_{214}\bar{z}^4+((3i\nu-1)H_{12}+(-5i+\nu)(2i+\nu)V_{123}V_{213} \\
&+(((2i-3\nu)\nu-34)V_{123}+(-5i+\nu)(2i+\nu)V_{124})V_{214})\bar{z}^3+(16H_{12}+2(2(\nu^2+4)V_{124}-3(\nu^2+6)V_{123})V_{213}+((9\nu^2+64)V_{123} \\
&-6(\nu^2+6)V_{124})V_{214})\bar{z}^2+(3(\nu(3i+\nu)+10)V_{123}-2(\nu(6i+\nu)+11)V_{124})V_{213}\bar{z}+(3(\nu(3i+\nu)+10)V_{124}-2(\nu(3i+2\nu)+19)V_{123})V_{214}\bar{z} \\
&+2(-2i+\nu)(-i+\nu)(V_{123}-V_{124})(V_{213}-V_{214}))z^2+\bar{z}(((2-2i\nu)H_{12}-2(i+\nu)(2i+\nu)V_{124}V_{214}+V_{123}((\nu(24i+5\nu)-24)V_{214} \\
&-2(i+\nu)(2i+\nu)V_{213}))\bar{z}^3+2((-i\nu-10)H_{12}+2(2i+\nu)V_{124}((4i+\nu)V_{214}-(i+\nu)V_{213})+2(4i+\nu)V_{123}((2i+\nu)V_{213} \\
&-(4i+\nu)V_{214}))\bar{z}^2+2(V_{124}(3(2i+\nu)(-5i+\nu)V_{214}-2(\nu(-6i+\nu)+11)V_{213})+V_{123}(3(2i+\nu)(-5i+\nu)V_{213}-2(\nu(-3i+2\nu)+19)V_{214}))\bar{z} \\
&-\bar{z}^4(2i+\nu)^2V_{123}V_{214}+8(\nu^2+4)(V_{123}-V_{124})(V_{214}-V_{213}))z-z^5(\bar{z}-1)(-2i+\nu)^2V_{123}V_{214}+\bar{z}^2((2i+\nu)^2V_{123}V_{214}\bar{z}^3+((i\nu+4)H_{12} \\
&+2(i+\nu)(2i+\nu)V_{123}V_{213}+((16-5\nu(4i+\nu))V_{123}+2(i+\nu)(2i+\nu)V_{124})V_{214})\bar{z}^2+2(i+\nu)(V_{123}(2(5i+2\nu)V_{214}-3(2i+\nu)V_{213}) \\
&+V_{124}(2(i+\nu)V_{213}-3(2i+\nu)V_{214}))\bar{z}+4(i+\nu)(2i+\nu)(V_{123}-V_{124})(V_{213}-V_{214}))).
\end{aligned}
\tag{A.5.4}
$$

Evaluating the ν integral by residues, with $\Delta_\psi \gg 1$, gives

$$
\begin{aligned}
\epsilon_\mu \tilde{\epsilon}_\nu \delta G^{\mu,\nu} = & \frac{2}{(z+\bar{z})^9} 120 i\pi z\bar{z}(\frac{1}{2} H_{12}(z+\bar{z})(24 n_f (z+\bar{z})(z^4 + 8z^3\bar{z} + 29 z^2\bar{z}^2 + 8z\bar{z}^3 + \bar{z}^4) + n_s(z^5(3\bar{z}+1) + z^4\bar{z}(32\bar{z}+5) \\
& + z^3\bar{z}^2(163\bar{z}-5) + z^2\bar{z}^3(32\bar{z}-5) + z\bar{z}^4(3\bar{z}+5) + \bar{z}^5)) - V_{124}V_{213}(8n_f(2z^6 + 3z^5(8\bar{z}-1) + z^4\bar{z}(138\bar{z}-35) - z^3\bar{z}^2(188\bar{z}+195) \\
& + 3z^2\bar{z}^3(46\bar{z}-65) + z\bar{z}^4(24\bar{z}-35) + \bar{z}^5(2\bar{z}-3)) - n_s(3z^5(4\bar{z}-3) + z^4(\bar{z}(168\bar{z}-145)+12) + 3z^3\bar{z}(384\bar{z}^2 - 375\bar{z} + 56) \\
& + 3z^2\bar{z}^2(\bar{z}(56\bar{z}-375)+384) + z\bar{z}^3(\bar{z}(12\bar{z}-145)+168) + 3\bar{z}^4(4-3\bar{z}))) + (z-1)(\bar{z}-1)V_{123}V_{214}(8n_f(2z^6 + 3z^5(8\bar{z}-1) \\
& + z^4\bar{z}(138\bar{z}-35) - z^3\bar{z}^2(188\bar{z}+195) + 3z^2\bar{z}^3(46\bar{z}-65) + z\bar{z}^4(24\bar{z}-35) + \bar{z}^5(2\bar{z}-3)) + n_s(4z^6 + 15z^5(4\bar{z}-1) + z^4(\bar{z}(444\bar{z}-215)+12) \\
& + z^3\bar{z}(\bar{z}(776\bar{z}-1515)+168) + 3z^2\bar{z}^2(\bar{z}(148\bar{z}-505)+384) + z\bar{z}^3(5\bar{z}(12\bar{z}-43)+168) + \bar{z}^4(\bar{z}(4\bar{z}-15)+12))) \\
& + 6n_s(z-1)(\bar{z}-1)V_{123}V_{213}(z+\bar{z}-2)(z^4 + 14z^3\bar{z} + 96z^2\bar{z}^2 + 14z\bar{z}^3 + \bar{z}^4) \\
& + 6n_s(z-1)(\bar{z}-1)V_{124}V_{214}(z+\bar{z}-2)(z^4 + 14z^3\bar{z} + 96z^2\bar{z}^2 + 14z\bar{z}^3 + \bar{z}^4)) \ .
\end{aligned} \tag{A.5.5}
$$

A.6 Three-point functions of conserved currents

In this appendix we summarize conventions used through out the chapter in describing the OPE coefficients appearing in the correlation functions of conserved currents. Correlation functions of conserved currents in CFT are derived in [6] (see also [167]) using conformal symmetry. Expression written here can be compared with similar ones written in [60, 61, 168–171].

A.6.1 $\langle JJT \rangle$

Two point function of spin-1 currents is given by

$$
\langle \varepsilon_1 . J(x_1) \varepsilon_2 . J(x_2) \rangle = c_J \frac{H_{12}}{x_{12}^{2d}} \ , \tag{A.6.1}
$$

where, H_{12} is defined in (A.13.1). The three-point function $\langle JJT \rangle$ is given by

$$
\langle J(x_1) J(x_2) T(x_3) \rangle = \frac{\alpha_1 V_1 V_2 V_3^2 + \alpha_2 H_{12} V_3^2 + \alpha_3 (H_{23} V_1 V_3 + H_{13} V_2 V_3) + \alpha_5 H_{13} H_{23}}{x_{12}^{d-2} x_{13}^{d-2} x_{23}^{d+2}} \tag{A.6.2}
$$

with

$$V_1 = V_{1,23}, \qquad V_2 = V_{2,31}, \qquad V_3 = V_{3,12}, \tag{A.6.3}$$

In the free field basis, this can also be written as

$$\langle JJT \rangle = n_s \langle JJT \rangle_{scalar} + n_f \langle JJT \rangle_{fermion} \tag{A.6.4}$$

where the coefficients are related by [64]

$$\begin{aligned} \alpha_1 &= n_s \frac{d-2}{2(d-1)} - 8n_f, \quad \alpha_2 = -4n_f - \frac{n_s}{2(d-1)} \\ \alpha_3 &= -4n_f - \frac{n_s}{d-1}, \quad \alpha_5 = \frac{n_s}{(d-1)(d-2)}. \end{aligned} \tag{A.6.5}$$

The Ward identity relates one combination of n_s and n_f to the two-point function:

$$c_J = \frac{S_d}{d} \left(4n_f + \frac{n_s}{d-2} \right) \ , \tag{A.6.6}$$

where, $S_D = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

A.6.2 $\langle TTT \rangle$

The central charge c_T is defined as

$$\langle \varepsilon_1 . T(x_1) \varepsilon_2 . T(x_2) \rangle = c_T \frac{H_{12}^2}{x_{12}^{2(d+2)}} \ , \tag{A.6.7}$$

where, H_{12} is given by equation (A.13.1).

Three point function $\langle T(x_1) T(x_2) T(x_3) \rangle$ is fixed by conformal invariance and permutation symmetry

$$\langle T(x_1) T(x_2) T(x_3) \rangle = \frac{\sum_{i=1}^{5} \alpha_i S_i}{x_{12}^{2+d} x_{13}^{2+d} x_{23}^{2+d}} \tag{A.6.8}$$

where

$$S_1 = V_1^2 V_2^2 V_3^2, \quad S_2 = V_1 V_2 V_3 \left(H_{23}V_1 + H_{13}V_2 + H_{12}V_3\right) \tag{A.6.9}$$
$$S_3 = \left(H_{12}H_{23}V_1V_3 + H_{13}V_2\left(H_{23}V_1 + H_{12}V_3\right)\right), \quad S_4 = H_{12}H_{13}H_{23}$$
$$S_5 = H_{23}^2 V_1^2 + H_{13}^2 V_2^2 + H_{12}^2 V_3^2.$$

This three-point function can be translated to the free-field basis

$$\langle TTT\rangle = \tilde{n}_s \langle TTT\rangle_{scalar} + \tilde{n}_f \langle TTT\rangle_{fermion} + \tilde{n}_v \langle TTT\rangle_{vector} \tag{A.6.10}$$

using [64]

$$\begin{aligned}
\alpha_1 &= 128d^2\tilde{n}_f - \frac{8d^2(d-2)^3}{(d-1)^3}\tilde{n}_s - 8192\tilde{n}_v \\
\alpha_2 &= 64d(d-2)\tilde{n}_f + \frac{32(d-2)^2d^2}{(d-1)^3}\tilde{n}_s - 8192\tilde{n}_v \\
\alpha_3 &= -128d\tilde{n}_f - \frac{64d^2(d-2)}{(d-1)^3}\tilde{n}_s - 4096\tilde{n}_v \\
\alpha_4 &= \frac{64d^2}{(d-1)^3}\tilde{n}_s - \frac{4096}{d-2}\tilde{n}_v \\
\alpha_5 &= -64d\tilde{n}_f - \frac{16d(d-2)^2}{(d-1)^3}\tilde{n}_s - 2048\tilde{n}_v \ .
\end{aligned} \tag{A.6.11}$$

Ward identity relates $\tilde{n}_s$, $\tilde{n}_f$, and $\tilde{n}_v$ to the central charge in the following way

$$c_T = 128S_d\left(\tilde{n}_f + \frac{1}{2(d-1)}\tilde{n}_s + \frac{16(d-3)}{d(d-2)}\tilde{n}_v\right) \ . \tag{A.6.12}$$

A.7 Three-point functions in $D = 3$

A.7.1 $\langle JJT \rangle$

The parity odd part of the correlation functions is given by [171]

$$\langle J(x_3)J(x_4)T(x_5)\rangle = n_{\rm odd}\frac{Q_3^2 S_3^2 + 2P_5 S_4^2 + 2P_4 S_5^2}{|x_{34}||x_{35}||x_{45}|}, \tag{A.7.1}$$

where,

$$\begin{aligned}
Q_3 &= \frac{2\varepsilon_3 \cdot x_{35}}{x_{35}^2} - \frac{2\varepsilon_3 \cdot x_{34}}{x_{34}^2},\\
Q_4 &= \frac{2\varepsilon_4 \cdot x_{43}}{x_{43}^2} - \frac{2\varepsilon_4 \cdot x_{45}}{x_{45}^2},\\
Q_5 &= \frac{2\varepsilon_5 \cdot x_{54}}{x_{54}^2} - \frac{2\varepsilon_5 \cdot x_{53}}{x_{53}^2},\\
P_3 &= \frac{4x_{34} \cdot \varepsilon_3 x_{34} \cdot \varepsilon_4}{(x_{34} \cdot x_{34})^2} - \frac{2\varepsilon_3 \cdot \varepsilon_4}{x_{34} \cdot x_{34}}\\
P_4 &= \frac{4x_{45} \cdot \varepsilon_4 x_{45} \cdot \varepsilon_5}{(x_{45} \cdot x_{45})^2} - \frac{2\varepsilon_4 \cdot \varepsilon_5}{x_{45} \cdot x_{45}}\\
P_5 &= \frac{4x_{53} \cdot \varepsilon_3 x_{53} \cdot \varepsilon_5}{(x_{53} \cdot x_{53})^2} - \frac{2\varepsilon_5 \cdot \varepsilon_3}{x_{53} \cdot x_{53}}\\
S_3^2 &= -\frac{2\left(x_{34}^2 \epsilon\left(x_{53}, \varepsilon_3, \varepsilon_4\right) + x_{53}^2 \epsilon\left(x_{34}, \varepsilon_3, \varepsilon_4\right) - 2\epsilon\left(x_{34}, x_{53}, \varepsilon_3\right)\varepsilon_4 \cdot x_{34}\right)}{|x_{34}|^3 |x_{45}||x_{53}|}\\
S_4^2 &= \frac{2\left(x_{34}^2 \epsilon\left(x_{45}, \varepsilon_4, \varepsilon_5\right) + x_{45}^2 \epsilon\left(x_{34}, \varepsilon_4, \varepsilon_5\right) - 2\epsilon\left(x_{45}, x_{34}, \varepsilon_4\right)\varepsilon_5 \cdot x_{45}\right)}{|x_{34}||x_{45}|^3 |x_{53}|}\\
S_5^2 &= -\frac{2\left(x_{45}^2 \epsilon\left(x_{53}, \varepsilon_5, \varepsilon_3\right) + x_{53}^2 \epsilon\left(x_{45}, \varepsilon_5, \varepsilon_3\right) - 2\epsilon\left(x_{53}, x_{45}, \varepsilon_5\right)\varepsilon_3 \cdot x_{53}\right)}{|x_{34}||x_{45}||x_{53}|^3},
\end{aligned} \tag{A.7.2}$$

where $\epsilon\,(a, b, c) \equiv \epsilon_{\mu\nu\alpha} a^\mu b^\nu c^\alpha$, with $\epsilon_{\mu\nu\alpha}$ denoting the Levi-Civita symbol. The parity even part is given by (A.6.2) with $D = 3$.

A.7.2 $\langle TTT \rangle$

The parity odd part of the correlation functions is given by [171]

$$\langle T(x_3)T(x_4)T(x_5)\rangle = n_{\rm odd}\frac{P_5 Q_5^2 S_5^2 + P_3\left(Q_3^2 S_3^2 - 5P_5 S_4^2\right) + P_4\left(5P_5 S_3^2 + 5P_3 S_5^2 - Q_4^2 S_4^2\right)}{|x_{34}||x_{45}||x_{53}|}, \tag{A.7.3}$$

where the structures are defined in the previous subsection. The parity even part is given by (A.6.8) with $D = 3$ and $\tilde{n}_v = 0$.

A.8 d-dimensional smearing integrals

We are interested in evaluating integrals of the form

$$\int d^{d-1}\vec{\mathbf{p}}\frac{\prod_i \vec{\mathbf{p}}.\vec{v}_i}{(\vec{\mathbf{p}}^2 + \vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_1}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_2}}. \tag{A.8.1}$$

Let us first define[2]

$$I_{p_1,p_2}(\vec{\mathbf{L}}) \equiv \int d^{d-1}\vec{\mathbf{p}}\frac{1}{(\vec{\mathbf{p}}^2 + \vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_1}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_2}}. \tag{A.8.2}$$

Using Feynman parametrization we can rewrite this as

$$I_{p_1,p_2}(\vec{\mathbf{L}}) = \frac{\Gamma(p_1+p_2)}{\Gamma(p_1)\Gamma(p_2)}\int_0^1 d\alpha \int d^{d-1}\vec{\mathbf{p}}\frac{\alpha^{p_1-1}(1-\alpha)^{p_2-1}}{(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}} + \alpha\vec{\mathbf{p}}\cdot\vec{\mathbf{p}})^{p_1+p_2}}. \tag{A.8.3}$$

The idea is to use derivatives with respect to $\vec{\mathbf{L}}$ to obtain an expression with powers of $\vec{\mathbf{p}}$ in the numerator. To this end, let us first define

$$\begin{aligned} K_p(\vec{\mathbf{L}}) &\equiv \int d^{d-1}\vec{\mathbf{p}}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}} + \alpha\vec{\mathbf{p}}\cdot\vec{\mathbf{p}})^{-p} \\ &= \frac{i^{d-1}\pi^{\frac{d-1}{2}}(-1)^p 2^{-d+2p+1}\alpha^{-d+p+1}\Gamma\left(\frac{1-d}{2}+p\right)(\vec{\mathbf{L}}\cdot\vec{\mathbf{L}})^{\frac{d-1}{2}-p}}{\Gamma(p)}. \end{aligned} \tag{A.8.4}$$

[2] note that $p_1, p_2 > 0$ in all expression appearing in this chapter.

Furthermore, let us notice that

$$\frac{(-1)^{-n}\Gamma(p-n)}{\Gamma(p)}\prod_i^n \frac{\partial}{\partial L^{\mu_i}}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}}+\alpha\vec{\mathbf{p}}\cdot\vec{\mathbf{p}})^{n-p} = (\vec{\mathbf{p}}\cdot L+\alpha\vec{\mathbf{p}}\cdot\vec{\mathbf{p}})^{-p}\prod_i^n p^{\mu_i}. \quad \text{(A.8.5)}$$

Finally we define

$$\begin{aligned} F^{(n)}_{p_1,p_2}(\vec{\mathbf{L}}) &\equiv \frac{\Gamma(p_1+p_2-n)(-1)^{-n}}{\Gamma(p_1)\Gamma(p_2)}\int_0^1 d\alpha\alpha^{p_1-1}(1-\alpha)^{p_2-1}K_{p_1+p_2-n}(\vec{\mathbf{L}}) \\ &= \frac{i^{d-1}\pi^{\frac{d-1}{2}}\Gamma\left(-\frac{d}{2}-n+p_1+p_2+\frac{1}{2}\right)\Gamma(-d-n+2p_1+p_2+1)(L\cdot L)^{\frac{d-1}{2}+n-p_1-p_2}}{(-1)^{2n-p_1-p_2}2^{d-2(-n+p_1+p_2)-1}\Gamma(p_1)\Gamma(-d-n+2(p_1+p_2)+1)}. \end{aligned} \quad \text{(A.8.6)}$$

Using this, we now have a simple way of evaluating integrals:

$$\int d^{d-1}\vec{\mathbf{p}}\frac{\prod_i^n \vec{\mathbf{p}}.\vec{v}_i}{(\vec{\mathbf{p}}^2+\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_1}(\vec{\mathbf{p}}\cdot\vec{\mathbf{L}})^{p_2}} = \prod_i^n(\vec{v}_i\cdot\vec{\partial})F^{(n)}_{p_1,p_2}(\vec{\mathbf{L}}), \quad \text{(A.8.7)}$$

where $\vec{\partial}$ signifies differentiation with respect to $\vec{\mathbf{L}}$.

A.9 Polarization vectors

Throughout this chapter, we used a particular null vector 4.3.14, to construct the polarization tensors corresponding to the external smeared states. The same null vector was used in [9] for obtaining $a=c$ bounds in $D=4$. In this appendix we will describe how this choice simplifies the task of extracting positivity conditions from spinning correlators with conserved operator insertions. For the case of non-conserved operators, this is not the most general choice of polarizations and does not necessarily lead to the most optimum bounds. However, the bounds obtained using this vector are sufficiently stringent for our purposes.

Conserved operators

Defining holographic operator $\mathcal{E}_r(v)$ requires choosing a null direction u, similar to the conformal collider setup in [13]. Let us call this d-dimensional vector $\hat{u} = (-1, \hat{n}) = (-1, 1, \vec{0})$ and denote $n^\mu = (0, 1, \vec{0})$. For most of the following discussion $D \geq 4$ and $D = 3$ is considered separately in the chapter.

We are interested in computing smeared spinning external states,

$$\epsilon_{1}{}^{\star}_{\alpha_1\alpha_2\cdots\alpha_{s_1}} \langle O_1(\omega)^{\alpha_1\alpha_2\cdots\alpha_s} \mathcal{E}_r(\nu) O_2(\omega)^{\beta_1\beta_2\cdots\beta_{s_2}} \rangle \epsilon_{2\beta_1\beta_2\cdots\beta_{s_2}}, \tag{A.9.1}$$

where $\star$ denotes complex conjugation. By smearing external operators, we are preparing states with definite momenta, $\omega^\mu = \omega t^\mu$ along the time direction with $t^2 = -1$. Primary operators considered here are in the symmetric traceless representations, so polarization tensors can be chosen to be symmetric and traceless. Conservation equation implies

$$\omega_{\mu_1} \langle O_1(\omega)^{\mu_1\mu_2\cdots\mu_{s_1}} \cdots \rangle = 0. \tag{A.9.2}$$

Therefore, we are free to choose ϵ with vanishing time-like components so that we have $\epsilon = \epsilon_{i_1\cdots i_{s_1}}$.

As a first example let us choose external state created by wave-packets of the stress tensor. The expectation value of holographic null energy operator has the following decomposition under $SO(d-1)$ corresponding to spatial rotations :

$$\langle \mathcal{E}_r(\nu) \rangle = \langle 0 | \epsilon^\star_{ij} T_{ij}(\omega) \mathcal{E}_r(\nu) \epsilon_{lk} T_{lk}(\omega) | 0 \rangle = \tilde{t}_0 \epsilon^\star_{ij} \epsilon_{ij} + \tilde{t}_2 \epsilon^\star_{ij} \epsilon_{il} \hat{n}_j \hat{n}_l + \tilde{t}_4 |\epsilon_{ij} \hat{n}_i \hat{n}_j|^2. \tag{A.9.3}$$

Using the positivity of this expectation value for any ϵ_{ij}, we look for the optimal bounds on coefficients. Following [13], we further decompose this expression in terms of irreducible representations, i.e. spin 0, 1, 2 under $SO(d-2)$, corresponding to rotations that leave the spatial part of the null direction $\hat{n}^i$ invariant. More explicitly, let us parametrize a purely spatial polarization tensor as[3]

$$\epsilon_{ij} = e_{ij} + b_{(i}\hat{n}_{j)} + \alpha\left(\hat{n}_i\hat{n}_j - \frac{\delta_{ij}}{d-1}\right), \tag{A.9.4}$$

where e_{ij} and b_i satisfy $b_i\hat{n}_i = 0, e_{ij}\hat{n}^j = 0, e_{ii} = 0$ and α is an arbitrary complex number.

Substituting this expression in (A.9.3) we find

$$\langle\mathcal{E}_r\rangle = |\alpha|^2\left(\tilde{t}_0\frac{d-2}{d-1} + \tilde{t}_2\frac{(d-2)^2}{(d-1)^2} + \tilde{t}_4\frac{(d-2)^2}{(d-1)^2}\right) + \frac{b_i b^{\star i}}{2}\left(\tilde{t}_0 + \frac{\tilde{t}_2}{2}\right) + \tilde{t}_0 e_{ij}e^{\star ij}, \tag{A.9.5}$$

where each term in this expression corresponds to an irreducible representation. Since these terms do not mix under $SO(d-2)$ rotations, positivity of the holographic null energy operator implies the positivity of each term separately.

We will now show that the powers of λ^2 in (3.5.1) and (3.5.7) are in one to one correspondence with these irreducible representations. To demonstrate this let us consider the following polarization vector,

$$\begin{aligned} \epsilon^\mu &= \hat{v}^\mu + \epsilon_\perp^\mu\,, \qquad \epsilon_\perp = (0,0,i\lambda,\lambda,\underbrace{0,\cdots,0}_{d-4}), \\ \hat{v} &= (1,1,\underbrace{0,\cdots,0}_{d-2}), \end{aligned} \tag{A.9.6}$$

where λ is an arbitrary real number. Contracting this null vector with external operator, $T_{\mu\nu}\epsilon^\mu\epsilon^\nu$ we find

$$\langle\mathcal{E}_r\rangle = g_0 + g_2\lambda^2 + g_4\lambda^4. \tag{A.9.7}$$

[3]Note that in writing this parametrization, we have chosen $D \geq 4$ as can be seen by the fact that if $d \leq 3$, then $e_{ij} = 0$ for a traceless tensor.

Note that $\epsilon^\mu\epsilon^\nu$ is not a purely spatial polarization tensor. Since only the spatial components contribute, we will use the symmetric traceless projector[4] $\mathcal{Q}_{\mu\nu}^{\alpha\beta}$ to convert $\epsilon^\mu\epsilon^\nu$ into a purely spatial traceless polarization tensor $\mathcal{E}^{\mu\nu}$:

$$
\begin{aligned}
&\mathcal{P}_{\mu\nu} = \eta_{\mu\nu} + t_\mu t_\nu \\
&\mathcal{Q}_{\mu\nu}^{\alpha\beta} = \frac{1}{2}\left(\mathcal{P}_\mu^\alpha \mathcal{P}_\nu^\beta + \mathcal{P}_\nu^\alpha \mathcal{P}_\mu^\beta\right) - \frac{1}{d-1}\mathcal{P}_{\mu\nu}\mathcal{P}^{\alpha\beta} \qquad \mathcal{Q}_\mu^{\mu\alpha\beta} = 0 \\
&\mathcal{E}^{\alpha\beta} \equiv Q_{\mu\nu}^{\alpha\beta}\epsilon^\mu\epsilon^\nu = \epsilon_\perp^{(\alpha}\epsilon_\perp^{\beta)} + \epsilon_\perp^{(\alpha}\hat{v}^{\beta)} + (\hat{v}\cdot t)\epsilon_\perp^{(\alpha}t^{\beta)} \\
&+ (\hat{v}^\alpha + (\hat{v}\cdot t)t^\alpha)\left(\hat{v}^\beta + (\hat{v}\cdot t)t^\beta\right) - \frac{(\hat{v}\cdot t)^2}{d-1}(\delta^{\alpha\beta} + t^\alpha t^\beta) \\
&\Rightarrow \quad \mathcal{E}^{ij} = \epsilon_\perp^{(i}\epsilon_\perp^{j)} + \epsilon_\perp^{(i}\hat{n}^{j)} + (\hat{n}^i\hat{n}^j - \frac{\delta_{ij}}{d-1}),
\end{aligned}
\tag{A.9.8}
$$

which has the form of the decomposition in A.9.4. Furthermore, $\epsilon_\perp^{(i}\epsilon_\perp^{j)}, \epsilon_\perp^{(i}\hat{n}^{j)}$ satisfy the same conditions as e^{ij} and $b^{(i}n^{j)}$. In addition, any expression involving $\epsilon_\perp$ is multiplied with a power of λ. Therefore, $\epsilon_\perp^{(i}\hat{n}^{j)}$ and $\epsilon_\perp^{(i}\epsilon_\perp^{j)}$ are multiplied with λ and λ^2 respectively. This implies that each powers of λ^2 are in one-to-one correspondence with irreducible representations under $SO(d-2)$ rotations and g_0, g_2, g_4 should be positive independently.

This construction is easily generalized to the case of conserved higher spin operators. To do so, one finds a symmetric traceless projection operator and acts on a polarization tensor of the form $\epsilon^{\mu_1}\epsilon^{\mu_2}\cdots\epsilon^{\mu_s}$ with

$$
\begin{aligned}
&\mathcal{Q}_{\mu_1\mu_2\cdots\mu_s}^{\nu_1\nu_2\cdots\nu_s} \sim \frac{1}{s!}\left(\mathcal{P}_{\mu_1}^{(\nu_1}\mathcal{P}_{\mu_2}^{\nu_2}\cdots\mathcal{P}_{\mu_s}^{\nu_s)} - \text{traces}\right), \\
&\mathcal{P}_{\mu_1}^{\nu_1}\epsilon_\perp^{\mu_1} = {\epsilon_\perp}^{\nu_1}, \qquad \mathcal{P}_{\mu_1}^{\nu_1}\hat{v}^{\mu_1} = n_1^\nu, \\
&\mathcal{Q}_{\mu_1\mu_2\cdots\mu_s}^{\nu_1\nu_2\cdots\nu_s}\epsilon^{\mu_1}\epsilon^{\mu_2}\cdots\epsilon^{\mu_s} \sim \epsilon_\perp^{(\nu_1}\cdots\epsilon_\perp^{\nu_s)} + n^{(\nu_1}\epsilon_\perp^{\nu_2}\cdots\epsilon_\perp^{\nu_s)}, \\
&+\cdots+\left(n^{\nu_1}n^{\nu_2}\cdots n^{\nu_s} - \text{traces}\right),
\end{aligned}
\tag{A.9.9}
$$

[4]Note that the expectation values in states created by smearing conserved operators are unchanged under the action of $\mathcal{Q}$ due to conservation.

corresponding to spin $0, 1, \cdots, s-1, s$ representations under $SO(d-2)$. Each term has a different number of $\epsilon_\perp$, therefore the coefficients associated to powers of λ are independent and should satisfy positivity constraints separately.

In summary, for conserved operators, polarization vectors defined in 4.3.14 result in the most general possible bounds in the holographic collider setup described here.

Non-conserved operators

For non-conserved operators, the use of longitudinal polarizations will result in more general constraints. The bounds in this chapter were obtained using $\epsilon^\mu = (1, -1, \vec{0})$ as the longitudinal polarization tensor. It would interesting to find polarization tensors that result in the most optimal bounds. A more systematic approach would be useful in obtaining bounds in the light-cone limit to ensure the most stringent possible constraints.

A.10 Transverse Polarizations

We construct the transverse polarization tensors used in section 4.2 explicitly. These polarization tensors have only component in transverse directions $x - y$ so they can be used in $D \geq 4$. Let us define

$$x^+ = x + iy \ , \qquad x^- = x - iy \ . \tag{A.10.1}$$

Let us consider following basis vectors

$$e^{+} = \frac{1}{\sqrt{2}}(\partial_x - i\partial_y) \ , \qquad e_{-} = \frac{1}{\sqrt{2}}(\partial_x + i\partial_y) \ ,$$
$$e^{+\mu}\partial_{b^\mu} = \frac{1}{\sqrt{2}}\partial_{b^+} \ , \qquad e^{-\mu}\partial_{b^\mu} = \frac{1}{\sqrt{2}}\partial_{b^-} \ , \tag{A.10.2}$$

where both of them are null vector. Also we have $e^+ \cdot e^- = 1$. Hence they can be used for constructing the transverse traceless polarization tensor $e^{\mu_1\mu_2\cdots\mu_s}$:

$$e^{(+)\mu_1\mu_2\cdots\mu_s} = e^{+\mu_1}e^{+\mu_2}\cdots e^{+\mu_s} \ , \qquad e^{(-)\mu_1\mu_2\cdots\mu_s} = e^{-\mu_1}e^{-\mu_2}\cdots e^{-\mu_s}. \tag{A.10.3}$$

These polarization tensors are not orthogonal to each other. They can be made orthogonal by taking the following linear combinations

$$e^{\oplus\mu_1\mu_2\cdots\mu_s} = \frac{1}{\sqrt{2}}\left(e^{(+)\mu_1\mu_2\cdots\mu_s} + e^{(-)\mu_1\mu_2\cdots\mu_s}\right) \ ,$$
$$e^{\otimes\mu_1\mu_2\cdots\mu_s} = \frac{i}{\sqrt{2}}\left(e^{(+)\mu_1\mu_2\cdots\mu_s} - e^{(-)\mu_1\mu_2\cdots\mu_s}\right) \ , \tag{A.10.4}$$

where they satisfy

$$e^{\oplus\mu_1\mu_2\cdots\mu_s}e^{\otimes}{}_{\mu_1\mu_2\cdots\mu_s} = 0 \ , \qquad e^{\oplus\mu_1\mu_2\cdots\mu_s}e^{\oplus}{}_{\mu_1\mu_2\cdots\mu_s} = e^{\otimes\mu_1\mu_2\cdots\mu_s}e^{\otimes}{}_{\mu_1\mu_2\cdots\mu_s} = 1 \ ,$$
$$e^{\oplus\mu_1\mu_2\cdots\mu_s}\ e^{\oplus}_{\mu_1\mu_2\cdots\mu_s\mu_{s+1}\cdots\mu_{s+j}} = \frac{1}{\sqrt{2}}e^{\oplus}{}_{\mu_{s+1}\mu_{s+2}\cdots\mu_{s+j}} \ ,$$
$$e^{\otimes\mu_1\mu_2\cdots\mu_s}\ e^{\otimes}_{\mu_1\mu_2\cdots\mu_s\mu_{s+1}\cdots\mu_{s+j}} = \frac{1}{\sqrt{2}}e^{\oplus}{}_{\mu_{s+1}\mu_{s+2}\cdots\mu_{s+j}} \ , \tag{A.10.5}$$

where s and j are positive numbers.

A.11 Phase Shift Computations

A Lemma

In order to get the bounds in the transverse plane, we can use a trick that will be used many times in this appendix. After plugging the polarization tensors for particles, we always find the following equation

$$I = e^{\mu_1\mu_2\cdots\mu_i\mu_{i+1}\cdots\mu_J} e_{\mu_1\mu_2\cdots\mu_i}{}^{\nu_{i+1}\nu_{i+2}\cdots\nu_{J'}} \partial_{b^{\mu_{i+1}}} \cdots \partial_{b^{\mu_J}} \partial_{b^{\nu_{i+1}}} \cdots \partial_{b^{\nu_{J'}}} \frac{1}{b^{D-4}} \ . \tag{A.11.1}$$

We would like to show that sign of I alternates by choosing different directions for $\vec{b}$ in the transverse plane.

Let us first consider $J \neq J'$, $J' = J + K$. We specify x^+, x^- to be two arbitrary directions in the transverse plane and the direction of the impact parameter $\vec{b}$ is picked in the same plane spanned by x^+, x^-. By using $e = e^{\oplus}$ we find

$$\begin{aligned} I =& 2^{-1+J+K/2} \left(\partial_{b^+}^K + \partial_{b^-}^K\right) \left(\partial_{b^+}\partial_{b^-}\right)^J \frac{1}{b^{D-4}} \\ &= 2^{J+K/2}(-1)^K \left[\left(\frac{D-4}{2}\right)_J\right]^2 \left(\frac{D-4}{2} + J\right)_K \frac{\cos(K\theta)}{b^{D-4+2J+K}} \end{aligned} \tag{A.11.2}$$

where $(a)_b \equiv \frac{\Gamma(a+b)}{\Gamma(a)}$ and θ is the angle between the vector $\vec{b}$ and the x-axis, where $x = \frac{1}{\sqrt{2}}(x^+ + x^-)$. This implies that rotating $\vec{b}$ with respect to x-axis changes the sign of I for $K \neq 0$.

If $K = 0$, both $e^{\oplus}$ and $e^{\otimes}$ yield the same sign for I, and we need to use polarizations having components in other transverse directions, therefore the following argument could not be applied to $D = 4$. For $D \geq 5$, we can separate another transverse coordinate z from x^+, x^- and after taking derivative we place the impact parameter $\vec{b}$ in x, y, z plane. These coordinates are enough for getting the bounds and we do not have to consider other transverse directions in for $D \geq 6$. Again by plugging $e = e^{\oplus}$, we find

$$I = 2^{-1+J}\frac{\cos(\theta)^{2J}}{b^{D-4+2J}}\left(\frac{\Gamma(\frac{D-4}{2}+J)}{\Gamma(\frac{D-4}{2})}\right)^2 {}_2F_1\left(-J,-J,\frac{D-4}{2},-\tan(\theta)^2\right)\ , \quad \text{(A.11.3)}$$

where θ is the angle between $\hat{z}$ and $\vec{b}$. For any integer value of J and D, the hypergeometric function in (A.11.3) is a polynomial in its variable, changing sign for both even and odd J.

Diagonal Element Between $\mathcal{E}_J$

We set $\mathcal{E}^{(3)\mu_1\mu_2\cdots\mu_J} = z_{3T}^{\mu_1} z_{3T}^{\mu_1}\cdots z_{3T}^{\mu_J}, \mathcal{E}^{(1)\mu_1\mu_2\cdots\mu_J} = z_{1T}^{\mu_1} z_{1T}^{\mu_1}\cdots z_{1T}^{\mu_J}$ and send $e^{\mu_1}e^{\mu_2}\cdots e^{\mu_J}\to e^{\mu_1\mu_2\cdots\mu_J}$. We also need to impose $e_3^{\mu_1\mu_2\cdots\mu_J} = (e_1^{\mu_1\mu_2\cdots\mu_J})^\dagger$ to have positivity. With this choice of polarization, only $\mathcal{A}_1,\cdots,\mathcal{A}_{J+1}$ contribute to phase shift and we write down the contribution of each vertex to the phase shift. Let us define $\tilde{\delta}(s,\vec{b}) = \frac{\pi^{D/2-2}}{\Gamma(\frac{D-4}{2})G_N s}\delta(s,\vec{b})$,

$$\tilde{\delta}(s,\vec{b})\Big|_{\mathcal{A}_i} = (-1)^{(i-1)}a_i e^{\mu_1\cdots\mu_{i-1}\mu_i\mu_{i+1}\cdots\mu_J}e^{\nu_1\cdots\nu_{i-1}}{}_{\mu_i\mu_{i+1}\cdots\mu_J}\partial_{b^{\mu_1}}\cdots\partial_{b^{\mu_{i-1}}}\partial_{b^{\nu_1}}\cdots\partial_{b^{\nu_{i-1}}}\frac{1}{|b|^{D-4}}. \quad \text{(A.11.4)}$$

In the small impact parameter limit, the term with the most negative powers of b dominates over other terms. As explained in the lemma A.11, choosing different direction for $\vec{b}$ for $D\geq 5$ changes the sign for each of these terms. Therefore, by applying the argument successively, we find

$$a_i = 0 \qquad 2\leq i\leq J+1. \quad \text{(A.11.5)}$$

Note that for a_1, there is no derivative and hence rotating direction of $\vec{b}$ does not change the sign of this term. Choosing e to be either $e^{\otimes}$ or $e^{\oplus}$ we find for $\mathcal{A}_1$ a manifestly positive contribution

$$\tilde{\delta}(s,\vec{b})^{\oplus}\Big|_{\mathcal{A}_1} = \tilde{\delta}(s,\vec{b})^{\otimes}\Big|_{\mathcal{A}_1} = \frac{a_1}{|b|^{D-4}} \ . \tag{A.11.6}$$

$\mathcal{E}_{J-1}$

We again set $\mathcal{E}^{(3)\mu_1\mu_2\cdots\mu_J} = \mathcal{E}^{(1)\mu_1\mu_2\cdots\mu_J} = \epsilon_L^{(\mu_1}\epsilon_T^{\mu_2}\epsilon_T^{\mu_2}\cdots\epsilon_T^{\mu_J)}$. In this case all the remaining vertices contribute to the phase shift and each vertex contribution is as follows

$$\begin{aligned}\tilde{\delta}(s,\vec{b})\Big|_{\mathcal{A}_{2J+1+K}} =& \frac{2(-1)^{i-1}}{m^2J^2}(a_{2J+1+K}-(J-K)a_{J+K+1}) \\ & \times e^{\mu_1\cdots\mu_i\mu_{i+1}\cdots\mu_J}e^{\nu_1\cdots\nu_i}{}_{\mu_{i+1}\cdots\mu_J}\partial_{b^{\mu_1}}\cdots\partial_{b^{\mu_i}}\partial_{b^{\nu_1}}\cdots\partial_{b^{\nu_i}}\frac{1}{|b|^{D-4}} \ ,\end{aligned} \tag{A.11.7}$$

which by taking b small and using the trick discussed in A.11 yields

$$a_{2J+1+K} = (J-K)a_{J+K+1} \qquad 2 \leq K \leq J-1 \ . \tag{A.11.8}$$

While at the $\frac{1}{b^{D-2}}$ order, $\mathcal{A}_1$ contributes and we find

$$a_{2J+2}-(J-1)a_{J+2} = -a_1\frac{J(J-1)}{2} \ . \tag{A.11.9}$$

Off-diagonal Components of $\mathcal{E}_J$ and $\mathcal{E}_{J-1}$

In order to impose constraints on $\mathcal{A}_{J+2},\mathcal{A}_{J+3},\cdots\mathcal{A}_{2J+1}$, we use $\mathcal{E}^{(1)} = \mathcal{E}_J$, $\mathcal{E}^{(3)} = \mathcal{E}_{J-1}$. Subsequently, we find the contribution due to each of remaining vertices

$$\tilde{\delta}(s,\vec{b})\Big|_{\mathcal{A}_{J+1+i}} = \frac{2(-1)^i}{Jm}a_{J+1+i}e^{\mu_1\cdots\mu_i\mu_{i+1}\cdots\mu_J}e^{\nu_2\cdots\nu_i}{}_{\mu_{i+1}\cdots\mu_J}\partial_{b^{\mu_1}}\cdots\partial_{b^{\mu_i}}\partial_{b^{\nu_2}}\cdots\partial_{b^{\nu_i}}\frac{1}{|b|^{D-4}} \tag{A.11.10}$$

impling that $a_{J+1+i}=0$. Using the diagonal elements in $\mathcal{E}_{J-1}$ we find

$$a_{J+1+i}=0 \qquad i=2,\cdots,J, \tag{A.11.11}$$

$$a_{2J+1+i}=0 \qquad i=2,\cdots,J-1. \tag{A.11.12}$$

However, the contribution from $\mathcal{A}_1$ is given by

$$\tilde{\delta}(s,\vec{b})\Big|_{\mathcal{A}_1}=\frac{2(-i)}{m}a_1 e^{\mu_1\mu_2\cdots\mu_{J-1}\mu_J}e_{\mu_1\mu_2\cdots\mu_{J-1}}\partial_{b^{\mu_J}}\frac{1}{|b|^{D-4}}\ . \tag{A.11.13}$$

Therefore, we find $a_{J+2}=J\ a_1$, $a_{2J+2}=\frac{J(J-1)}{2}a_1$. This proves (4.2.26).

Diagonal Elements of $\mathcal{E}_{J-2}$

For constraining a_1 we used the diagonal elements in $\mathcal{E}_{J-2}$ for both particles. Computing C_{JJ2} after imposing all the other constraints, we find for $J\geq 4$

$$\tilde{\delta}(s,\vec{b})=a_1\frac{3(J-2)(J-3)}{m^4J(J-1)}\left(\frac{D+2J-6}{D+2J-5}\right)^2 e^{\mu_1\mu_2\mu_3\cdots\mu_{J-2}}e_{\mu_3\cdots\mu_{J-2}}{}^{\nu_1\nu_2}\partial_{b^{\mu_1}}\partial_{b^{\mu_2}}\partial_{b^{\nu_1}}\partial_{b^{\nu_2}}\frac{1}{|b|^{D-4}} \tag{A.11.14}$$

and hence $a_1=0$ due to the trick used in A.11. The equation A.11.14 is valid for $J\geq 4$. For $J=3$, we used interference between $\mathcal{E}^{(1)}=\mathcal{E}_0$ and $\mathcal{E}^{(3)}=\mathcal{E}_3$ to set $a_1=0$.

Bounds for $D = 4$

Positivity of the phase shift (4.2.36) leads to the following constraints in $D = 4$:

$$\begin{aligned}
&\bar{a}_n = 0 \ , \qquad n = 1, \cdots, 2J \ , \\
&\frac{a_{n+1}}{a_n} = \frac{(n-J)(n+J-1)}{n(2n-1)} \frac{1}{m^2} \ , \qquad n = 1, \cdots, J \ , \\
&\frac{a_{J+n+2}}{a_{J+n+1}} = \frac{n^2 - J^2}{n(2n+1)} \frac{1}{m^2} \ , \qquad n = 1, \cdots, J-1 \ ,
\end{aligned} \tag{A.11.15}$$

with $a_{J+2} = J a_1$.

A.12 Parity Violating Interactions in $D = 5$

Only in $D = 4$ and 5, the massive higher spin particles can interact with gravity in a way that violates parity. We already discussed the case of $D = 4$. Let us now discuss the parity odd interactions in $D = 5$. Unlike $D = 4$, only massive particles are allowed to couple to gravity in a way that does not preserve parity. In order to list all possible parity odd vertices for the interaction $J - J - 2$, we introduce the following parity odd building block:

$$\mathcal{B} = \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5} z_{1;\mu_1} z_{3;\mu_2} z_{\mu_3} q_{\mu_4} p_{3;\mu_5} \ . \tag{A.12.1}$$

The most general form of parity odd on-shell three-point amplitude can then be constructed using this building block. In particular, we can write two distinct sets of vertices. The first set contains J independent structures:

$$\begin{aligned}
\mathcal{A}_1^{odd} &= \mathcal{B}(z\cdot p_3)(z_1\cdot z_3)^{J-1}\ , \\
\mathcal{A}_2^{odd} &= \mathcal{B}(z\cdot p_3)(z_1\cdot z_3)^{J-2}(z_3\cdot q)(z_1\cdot q)\ , \\
&\vdots \\
\mathcal{A}_J^{odd} &= \mathcal{B}(z\cdot p_3)(z_3\cdot q)^{J-1}(z_1\cdot q)^{J-1}\ .
\end{aligned} \tag{A.12.2}$$

While the second set contains $J-1$ independent structures:

$$\begin{aligned}
\mathcal{A}_{J+1}^{odd} &= \mathcal{B}((z\cdot z_3)(z_1\cdot q)-(z\cdot z_1)(z_3\cdot q))(z_1\cdot z_3)^{J-2}, \\
\mathcal{A}_{J+2}^{odd} &= \mathcal{B}((z\cdot z_3)(z_1\cdot q)-(z\cdot z_1)(z_3\cdot q))(z_1\cdot z_3)^{J-3}(z_3\cdot q)(z_1\cdot q), \\
&\vdots \\
\mathcal{A}_{2J-1}^{odd} &= \mathcal{B}((z\cdot z_3)(z_1\cdot q)-(z\cdot z_1)(z_3\cdot q))(z_3\cdot q)^{J-2}(z_1\cdot q)^{J-2}\ .
\end{aligned} \tag{A.12.3}$$

The most general form of the parity violating three-point amplitude is given by

$$C_{JJ2} = \sqrt{32\pi G_N}\sum_{n=1}^{2J-1}\bar{a}_n\mathcal{A}_n^{odd}\ . \tag{A.12.4}$$

Bounds on parity violating interactions can be obtained by using a simple null polarization vector

$$\epsilon^\mu(p_1) = i\epsilon_L^\mu(p_1) - i\epsilon_{T,\hat{x}}^\mu(p_1) + \sqrt{2}\epsilon_{T,\hat{y}}^\mu(p_1)\ , \qquad \epsilon^\mu(p_3) = -i\epsilon_L^\mu(p_3) + i\epsilon_{T,\hat{x}}^\mu(p_3) + \sqrt{2}\epsilon_{T,\hat{y}}^\mu(p_3)\ , \tag{A.12.5}$$

where the transverse and longitudinal vectors are defined in (4.2.16). The vectors $\hat{x}$ and $\hat{y}$ are given by $\hat{x}=(0,0,1,0,0)$ and $\hat{y}=(0,0,0,1,0)$. Positivity of the phase shift for this polarization leads to

$$\bar{a}_n = 0\ , \qquad n = 1,\cdots,2J-1 \tag{A.12.6}$$

for any spin J. Note that this bound holds even for $J=1$ and 2.

A.13 Correlators of Higher Spin Operators in CFT

Let us first define the building blocks

$$H_{ij} \equiv x_{ij}^2 \varepsilon_i \cdot \varepsilon_j - 2(x_{ij} \cdot \varepsilon_i)(x_{ij} \cdot \varepsilon_j), \qquad V_{i,jk} \equiv \frac{x_{ij}^2 x_{ik} \cdot \varepsilon_i - x_{ik}^2 x_{ij} \cdot \varepsilon_i}{x_{jk}^2} \ , \quad \text{(A.13.1)}$$

where, $x_{ij}^\mu = (x_i - x_j)^\mu$.

Two-point function

$$\langle \varepsilon_1.X_\ell(x_1) \varepsilon_2.X_\ell(x_2) \rangle = C_{X_\ell} \frac{H_{12}^\ell}{x_{12}^{2(\Delta+\ell)}} \ , \qquad \text{(A.13.2)}$$

where, Δ is the dimension of the operator X_ℓ and C_{X_ℓ} is a positive constant. ε_1 and ε_2 are null polarization vectors contracted with the indices of X_ℓ in the following way

$$(\varepsilon^\mu \varepsilon^\nu \cdots)\, X_{\mu\nu\cdots} \equiv \varepsilon.X \ . \qquad \text{(A.13.3)}$$

Three-point Function

Let us now discuss the three-point function $\langle \varepsilon_1.X_\ell(x_1) \varepsilon_2.X_\ell(x_2) \varepsilon_3.T(x_3) \rangle$:

$$\begin{aligned}&\langle \varepsilon_1.X_\ell(x_1) \varepsilon_2.X_\ell(x_2) \varepsilon_3.T(x_3) \rangle \\ &\qquad = \sum_{\{n_{23}, n_{13}, n_{12}\}} C_{n_{23}, n_{13}, n_{12}} \frac{V_{1,23}^{\ell-n_{12}-n_{13}} V_{2,13}^{\ell-n_{12}-n_{23}} V_{3,12}^{2-n_{13}-n_{23}} H_{12}^{n_{12}} H_{13}^{n_{13}} H_{23}^{n_{23}}}{x_{12}^{(2h-d-2)} x_{13}^{(d+2)} x_{23}^{(d+2)}},\end{aligned} \qquad \text{(A.13.4)}$$

where $C_{n_{23}, n_{13}, n_{12}}$ are OPE coefficients and $h \equiv \Delta + \ell$. In the above expression all of the polarization vectors are null, however polarizations $\varepsilon^\mu \varepsilon^\nu \cdots$ can be converted into an arbitrary polarization tensor $\varepsilon^{\mu\nu\cdots}$ by using projection operators from [60].

The sum in (A.13.4) is over all triplets of non-negative integers $\{n_{23}, n_{13}, n_{12}\}$ satisfying

$$\ell - n_{12} - n_{13} \geq 0 \ , \qquad \ell - n_{12} - n_{23} \geq 0 \ , \qquad 2 - n_{13} - n_{23} \geq 0 \ . \qquad \text{(A.13.5)}$$

To begin with, there are $5+6(\ell-1)$ OPE coefficients $C_{n_{23},n_{13},n_{12}}$, however, not all of them are independent. The three-point function (A.13.4) must be symmetric with respect to the exchange $(x_1, \varepsilon_1) \leftrightarrow (x_2, \varepsilon_2)$ which implies that only 4ℓ OPE coefficients can be independent in general. Moreover, conservation of the stress-tensor operator T will impose additional restrictions on the remaining OPE coefficients $C_{n_{23},n_{13},n_{12}}$.

Conservation Equation

Relations between the OPE coefficients from conservation of the stress-tensor operator T can be obtained by imposing the vanishing of $\frac{\partial}{\partial x^\mu}\langle T(x)\cdots\rangle$ up to contact terms. For $\langle X_\ell X_\ell T\rangle$, the conservation equation leads to ℓ additional constraint amongst the remaining 4ℓ OPE coefficients. Therefore, the three-point function $\langle X_\ell X_\ell T\rangle$ is fixed by conformal invariance up to 3ℓ independent OPE coefficients. Furthermore, the Ward identity leads to a relation between these OPE coefficients and the coefficient of the two-point function C_{X_ℓ}.

A.14 Details of Spin-3 Calculation in $D > 4$

Constraints from Conservation Equation

Conservation equation leads to 3 relations among the OPE coefficients

$$C_{0,0,0} = -\frac{1}{3}\left(d^2+4d\right)C_{0,2,0} - \frac{1}{6}\left(-d^2-4d+12\right)C_{1,1,0} + 2C_{0,1,0}, \tag{A.14.1}$$

$$\begin{aligned} C_{0,0,1} = &-\frac{1}{2}\left(d^2+2d\right)C_{0,2,1} - \frac{1}{4}\left(-d^2-2d+8\right)C_{1,1,1} - \frac{3}{2}dC_{0,2,0} \\ &-\frac{1}{2}(2-d)C_{1,1,0} + 2C_{0,1,1}, \end{aligned} \tag{A.14.2}$$

$$C_{0,0,2} = -\frac{1}{2}\left(4-d^2\right)C_{1,1,2} - 2dC_{0,2,1} - \frac{1}{2}(2-d)C_{1,1,1} + 2C_{0,1,2}\ . \tag{A.14.3}$$

Deriving Constraints from the HNEC

Let us first start with $\xi = +1$. In the limit $\rho \to 1$, the leading contribution to $\mathbf{E}(\rho)$ goes as $(1-\rho)^{-(d+3)}$, in particular

$$\mathbf{E}_+(\rho) = \frac{d(-4+d^2) - 18d(2+d)\lambda^2 + 72(2+d)\lambda^4 - 48\lambda^6}{(1-\rho)^{d+3}}t_1 + \cdots \tag{A.14.4}$$

up to some overall positive coefficient. t_1 in the above expression is a particular linear combination of all the OPE coefficients. Positivity of coefficients of each powers of λ^2 leads to the constraint

$$t_1 = 0\ . \tag{A.14.5}$$

After imposing this constraint, the next leading term becomes

$$\mathbf{E}_+(\rho) = \frac{(d-2)d - 12d\lambda^2 + 24\lambda^4}{(1-\rho)^{d+2}}t_2 + \cdots\ , \tag{A.14.6}$$

where, t_2 is another linear combination of all the OPE coefficients. Positivity now implies

$$t_2 = 0 \ . \tag{A.14.7}$$

After imposing both these constraints the next leading contribution can be written in terms of two new linear combinations t_3 and t_4 of OPE coefficients:

$$\mathbf{E}_+(\rho) = \frac{t_3 - (a_3 t_3 + a_4 t_4)\lambda^2 + t_4\lambda^4 - (b_3 t_3 + b_4 t_4)\lambda^6}{(1-\rho)^{d+1}} + \cdots \ , \tag{A.14.8}$$

where, a_3, a_4, b_3, b_4 are numerical factors shown later in this appendix. The exact values of these numerical factors are not important, but note that $a_3, a_4, b_4 > 0$ for $d > 3$. Positivity of coefficients of λ^0 and λ^4 imply that $t_3, t_4 \geq 0$. Then, positivity of coefficients of λ^2 dictates that

$$t_3 = t_4 = 0 \ . \tag{A.14.9}$$

After imposing these constraints, we get something very similar

$$\mathbf{E}_+(\rho) = \frac{t_5 - (a_5 t_5 + a_6 t_6)\lambda^2 + t_6\lambda^4}{(1-\rho)^{d}} + \cdots \ , \tag{A.14.10}$$

where, t_5 and t_6 are two new linear combinations of OPE coefficients and a_5, a_6 are positive numerical factors shown at the end of this appendix. Note that there is no λ^6 term in this order. However, positivity of coefficients of λ^0, λ^2 and λ^4 still produces two equalities:

$$t_5 = t_6 = 0 \ . \tag{A.14.11}$$

Repeating the same procedure for the next order, we obtain

$$\mathbf{E}_+(\rho) = \frac{t_7 - (a_7 t_7 + a_8 t_8)\lambda^2 + t_8\lambda^4 - (b_7 t_7 + b_8 t_8)\lambda^6}{(1-\rho)^{d-1}} + \cdots \ , \tag{A.14.12}$$

where, a and b coefficients are shown at the end of this appendix. A similar argument in $D \geq 4$ leads to constraints

$$t_7 = t_8 = 0 \ . \tag{A.14.13}$$

After imposing all these constraints, finally we obtain

$$\mathbf{E}_{+}(\rho)=\frac{t_9}{(1-\rho)^{d-2}}\left(1+\frac{4\Delta\lambda^2}{-d+2\Delta-2}+\frac{4\Delta(\Delta+1)\lambda^4}{(d-2\Delta)(d-2\Delta+2)}\right)\ , \quad \text{(A.14.14)}$$

where, coefficients of λ^0, λ^2 and λ^4 are now all positive. Hence, the holographic null energy condition now leads to $t_9 \geq 0$. We can now choose $\xi = -1$ and calculate $\mathbf{E}_{-}(\rho)$. After imposing $t_i = 0$ for $i = 1, \cdots, 8$, we get

$$\mathbf{E}_{-}(\rho)=-\frac{t_9}{(1-\rho)^{d-2}}\left(1+\frac{4\Delta\lambda^2}{-d+2\Delta-2}+\frac{4\Delta(\Delta+1)\lambda^4}{(d-2\Delta)(d-2\Delta+2)}\right) \quad \text{(A.14.15)}$$

and hence $t_9 \leq 0$. Therefore, combining both these inequalities, we finally get

$$t_9 = 0\ . \quad \text{(A.14.16)}$$

From the definitions of t_i's it is apparent that $t_1, \cdots, t_9$ are independent linear combinations of the OPE coefficients. Therefore, irrespective of their exact structures, $\{t_1, \cdots, t_9\}$ forms a complete basis in the space of OPE coefficients. As a consequence, the constraints $t_1, \cdots, t_9 = 0$ necessarily require that all OPE coefficients $C_{i,j,k}$ must vanish.

a and b Coefficients

a and b coefficients are given by

$$a_3 = \frac{2d(13\Delta+9)-8(\Delta+3)}{(d-2)(d(4\Delta+3)-2(\Delta+2))}\,, \qquad b_3 = -\frac{16\Delta}{(d-2)d(d(4\Delta+3)-2(\Delta+2))}\,,$$
$$a_4 = \frac{(d-3)d(\Delta+1)}{8d\Delta+6d-4\Delta-8}\,, \qquad b_4 = \frac{4\Delta+2}{4d\Delta+3d-2\Delta-4}\,,$$
$$a_5 = \frac{6(\Delta-1)}{(d-2)(2\Delta-1)}\,, \qquad a_6 = \frac{(d-3)\Delta}{2(2\Delta-1)}\,,$$
$$a_7 = \frac{2d^2(2\Delta(\Delta+1)(\Delta+2)-3)-4d(\Delta(\Delta+2)(7\Delta+1)-3)+48(\Delta+1)\Delta^2}{(d-2)(d-2\Delta)\left(2(d-5)\Delta^2+4(d-1)\Delta-3d+6\right)}\,,$$
$$b_7 = -\frac{8\Delta^2(\Delta+1)}{(d-2)(d-2\Delta)\left(2(d-5)\Delta^2+4(d-1)\Delta-3d+6\right)}\,,$$
$$a_8 = \frac{(d-3)(\Delta-1)(d-2(\Delta+1))}{2(d-5)\Delta^2+4(d-1)\Delta-3d+6}\,, \quad b_8 = \frac{2\left(2\Delta^2+\Delta-1\right)}{2(d-5)\Delta^2+4(d-1)\Delta-3d+6}\,.$$

t-basis in $D = 4$

For the purpose of illustration, let us transcribe $t_1, \cdots, t_9$ for $D = 4$. We will not show the general D expressions because the exact structures of $t_1, \cdots, t_9$ are not important. The fact that $t_1, \cdots, t_9$ are independent linear combinations of $C_{i,j,k}$

is sufficient to rule out the existence of spin-3 operators.

$$
\begin{aligned}
t_1 = &-\frac{5\pi^{7/2}4^{1-\Delta}\Gamma\left(\Delta-\frac{1}{2}\right)}{\Delta\left(\Delta^2-1\right)\Gamma(\Delta+4)}\{-(2\Delta+5)((\Delta+5)((\Delta+5)\Delta+28)\Delta+168)C_{0,1,0}+24(2\Delta+5)((\Delta+5)\Delta+10)C_{0,1,1}\\
&+\Delta(2(((((\Delta+17)\Delta+119)\Delta+471)\Delta+1044)\Delta+1156)C_{0,2,0}-24(((3\Delta+34)\Delta+121)\Delta+170)C_{0,2,1}\\
&-\Delta(((((\Delta+13)\Delta+91)\Delta+379)\Delta+964)C_{1,1,0}-12((3\Delta+26)\Delta+103)C_{1,1,1}+864C_{1,1,2})-576C_{0,1,2}\\
&-8(173C_{1,1,0}-300C_{1,1,1}+468C_{1,1,2}))+864C_{0,0,3}-48(30C_{0,1,2}-17C_{0,2,0}+22C_{0,2,1}+18C_{1,1,0}-39C_{1,1,1}+114C_{1,1,2})\}\ ,\\
t_2 = &\frac{5\pi^{7/2}2^{1-2\Delta}(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{3(\Delta-1)\Delta\left(3\Delta^4+26\Delta^3+103\Delta^2+200\Delta+156\right)\Gamma(\Delta+3)}\{-6\Delta^9C_{0,2,0}+3\Delta^9C_{1,1,0}-102\Delta^8C_{0,2,0}\\
&+45\Delta^8C_{1,1,0}-828\Delta^7C_{0,2,0}+334\Delta^7C_{1,1,0}+72\Delta^6C_{0,1,1}-4156\Delta^6C_{0,2,0}-288\Delta^6C_{0,2,1}+1562\Delta^6C_{1,1,0}\\
&+864\Delta^5C_{0,1,1}-14446\Delta^5C_{0,2,0}-432\Delta^5C_{0,2,1}+5067\Delta^5C_{1,1,0}-2592\Delta^5C_{1,1,2}+4584\Delta^4C_{0,1,1}-2592\Delta^4C_{0,1,2}\\
&-36662\Delta^4C_{0,2,0}+9888\Delta^4C_{0,2,1}+11773\Delta^4C_{1,1,0}-21600\Delta^4C_{1,1,2}+13632\Delta^3C_{0,1,1}-18432\Delta^3C_{0,1,2}\\
&-67616\Delta^3C_{0,2,0}+55920\Delta^3C_{0,2,1}+19292\Delta^3C_{1,1,0}-79200\Delta^3C_{1,1,2}+24816\Delta^2C_{0,1,1}-53856\Delta^2C_{0,1,2}\\
&-85464\Delta^2C_{0,2,0}+129408\Delta^2C_{0,2,1}+21108\Delta^2C_{1,1,0}-156960\Delta^2C_{1,1,2}+1728\left(3\Delta^3+15\Delta^2+35\Delta+30\right)C_{0,0,3}\\
&+\left(3\Delta^8+40\Delta^7+236\Delta^6+762\Delta^5+1393\Delta^4+1190\Delta^3-720\Delta^2-3024\Delta-1728\right)C_{0,1,0}+27072\Delta C_{0,1,1}\\
&-77760\Delta C_{0,1,2}-67392\Delta C_{0,2,0}+157824\Delta C_{0,2,1}+13968\Delta C_{1,1,0}-184896\Delta C_{1,1,2}+12096C_{0,1,1}-41472C_{0,1,2}\\
&-25920C_{0,2,0}+86400C_{0,2,1}+4320C_{1,1,0}-103680C_{1,1,2}\}\ ,\\
t_3 = &\frac{-\pi^{7/2}4^{-\Delta}(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{3(\Delta-1)\left(3\Delta^9+51\Delta^8+414\Delta^7+2078\Delta^6+7223\Delta^5+18331\Delta^4+33808\Delta^3+42732\Delta^2+33696\Delta+12960\right)\Gamma(\Delta+2))}\\
&\times\{1728(18\Delta^7+177\Delta^6+831\Delta^5+2334\Delta^4+4645\Delta^3+6783\Delta^2+5732\Delta+1680)C_{0,0,3}+(3\Delta^{12}+42\Delta^{11}+219\Delta^{10}\\
&+206\Delta^9-3651\Delta^8-24138\Delta^7-81903\Delta^6-183990\Delta^5-316308\Delta^4-452936\Delta^3-445512\Delta^2-140544\Delta+126144)C_{0,1,0}\\
&-2(3\Delta^{12}C_{1,1,0}+42\Delta^{11}C_{1,1,0}+432\Delta^{10}C_{0,2,1}+285\Delta^{10}C_{1,1,0}+720\Delta^9C_{0,2,1}+1448\Delta^9C_{1,1,0}+5184\Delta^9C_{1,1,2}\\
&-31608\Delta^8C_{0,2,1}+6519\Delta^8C_{1,1,0}+62208\Delta^8C_{1,1,2}-264672\Delta^7C_{0,2,1}+24066\Delta^7C_{1,1,0}+340416\Delta^7C_{1,1,2}\\
&-1033008\Delta^6C_{0,2,1}+67035\Delta^6C_{1,1,0}+1107648\Delta^6C_{1,1,2}-2495520\Delta^5C_{0,2,1}+140208\Delta^5C_{1,1,0}+2474496\Delta^5C_{1,1,2}\\
&-4233192\Delta^4C_{0,2,1}+220446\Delta^4C_{1,1,0}+4206816\Delta^4C_{1,1,2}-5473296\Delta^3C_{0,2,1}+264508\Delta^3C_{1,1,0}+5894208\Delta^3C_{1,1,2}\\
&-5511264\Delta^2C_{0,2,1}+234480\Delta^2C_{1,1,0}+6862752\Delta^2C_{1,1,2}+432(15\Delta^8+172\Delta^7+888\Delta^6+2690\Delta^5+5447\Delta^4\\
&+8078\Delta^3+7918\Delta^2+3304\Delta-624)C_{0,1,2}-12(9\Delta^{10}+141\Delta^9+1011\Delta^8+4350\Delta^7+12601\Delta^6+26427\Delta^5+43243\Delta^4\\
&+58314\Delta^3+53728\Delta^2+15312\Delta-14112)C_{0,1,1}-3664512\Delta C_{0,2,1}+129600\Delta C_{1,1,0}+5173632\Delta C_{1,1,2}-967680C_{0,2,1}\\
&+34560C_{1,1,0}+1347840C_{1,1,2})\}\ ,
\end{aligned}
$$

$$t_4 = \frac{\pi^{7/2}4^{-\Delta}(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{3(\Delta-1)\Delta\left(3\Delta^9+51\Delta^8+414\Delta^7+2078\Delta^6+7223\Delta^5+18331\Delta^4+33808\Delta^3+42732\Delta^2+33696\Delta+12960\right)\Gamma(\Delta+2)}$$

$$\times\{-1728\Delta\left(9\Delta^7+105\Delta^6+550\Delta^5+1797\Delta^4+4019\Delta^3+5976\Delta^2+4704\Delta+1152\right)C_{0,0,3}$$

$$-2(-144(33\Delta^9+482\Delta^8+3220\Delta^7+13428\Delta^6+39443\Delta^5+84574\Delta^4+129300\Delta^3+133632\Delta^2+88992\Delta+31104)C_{0,1,2}$$

$$+12(15\Delta^{11}+300\Delta^{10}+2836\Delta^9+16806\Delta^8+70033\Delta^7+217146\Delta^6+511924\Delta^5+913140\Delta^4+1197048\Delta^3+1090080\Delta^2+634176\Delta$$

$$+186624)C_{0,1,1}+\Delta(-288(9\Delta^9+135\Delta^8+902\Delta^7+3736\Delta^6+10842\Delta^5+22703\Delta^4+33325\Delta^3+32796\Delta^2+17496\Delta+1728)C_{1,1,2}$$

$$-24(30\Delta^{10}+411\Delta^9+2444\Delta^8+8520\Delta^7+19136\Delta^6+25089\Delta^5+1406\Delta^4-65772\Delta^3-129792\Delta^2-107712\Delta-27648)C_{0,2,1}$$

$$+(9\Delta^{12}+166\Delta^{11}+1543\Delta^{10}+9146\Delta^9+38267\Delta^8+119030\Delta^7+280469\Delta^6+495754\Delta^5+634144\Delta^4+536256\Delta^3$$

$$+238752\Delta^2-41472)C_{1,1,0}))+(9\Delta^{13}+208\Delta^{12}+2517\Delta^{11}+20148\Delta^{10}+116751\Delta^9+511632\Delta^8+1737543\Delta^7$$

$$+4628948\Delta^6+9669660\Delta^5+15584136\Delta^4+18714816\Delta^3+15761088\Delta^2+8439552\Delta+2239488)C_{0,1,0}\}\ ,$$

$$t_5 = \frac{\pi^{7/2}4^{-\Delta}(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{3(9\Delta^6+87\Delta^5+370\Delta^4+951\Delta^3+1667\Delta^2+1980\Delta+1008)\Gamma(\Delta+1)}\{(15\Delta^8+125\Delta^7+636\Delta^6+2162\Delta^5+5397\Delta^4$$

$$+9413\Delta^3+12150\Delta^2+10062\Delta+4212)C_{0,1,0}-2(15\Delta^8C_{1,1,0}+80\Delta^7C_{1,1,0}+258\Delta^6C_{1,1,0}+583\Delta^5C_{1,1,0}$$

$$+2592\Delta^5C_{1,1,2}+1130\Delta^4C_{1,1,0}+9936\Delta^4C_{1,1,2}+1317\Delta^3C_{1,1,0}+12096\Delta^3C_{1,1,2}+1333\Delta^2C_{1,1,0}-6480\Delta^2C_{1,1,2}$$

$$+6(9\Delta^6+84\Delta^5+400\Delta^4+988\Delta^3+1387\Delta^2+1036\Delta+384)C_{0,1,1}-12(18\Delta^6+285\Delta^5+1136\Delta^4+1817\Delta^3+752\Delta^2$$

$$-400\Delta-168)C_{0,2,1}+252\Delta C_{1,1,0}-6912\Delta C_{1,1,2}-720C_{1,1,0}+8208C_{1,1,2})\}\ ,$$

$$t_6 = \frac{-\pi^{7/2}2^{1-2\Delta}(\Delta+1)(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{(\Delta-1)\left(9\Delta^6+87\Delta^5+370\Delta^4+951\Delta^3+1667\Delta^2+1980\Delta+1008\right)\Gamma(\Delta+1)}\{(3\Delta^8+28\Delta^7+160\Delta^6+603\Delta^5$$

$$+1622\Delta^4+3005\Delta^3+4191\Delta^2+3564\Delta+1296)C_{0,1,0}-2(3\Delta^8C_{1,1,0}+19\Delta^7C_{1,1,0}+73\Delta^6C_{1,1,0}+173\Delta^5C_{1,1,0}$$

$$+327\Delta^4C_{1,1,0}-864\Delta^4C_{1,1,2}+354\Delta^3C_{1,1,0}-2016\Delta^3C_{1,1,2}+263\Delta^2C_{1,1,0}-4320\Delta^2C_{1,1,2}-12(6\Delta^4+47\Delta^3$$

$$+104\Delta^2+131\Delta+72)\Delta^2C_{0,2,1}+6(3\Delta^6+28\Delta^5+112\Delta^4+244\Delta^3+393\Delta^2+372\Delta+144)C_{0,1,1}$$

$$+12\Delta C_{1,1,0}-576\Delta C_{1,1,2}-144C_{1,1,0}+3456C_{1,1,2})\}\ ,$$

$$t_7 = \frac{\pi^{7/2}4^{-\Delta-1}\left(2\Delta^2-7\Delta+6\right)\Gamma\left(\Delta-\frac{3}{2}\right)}{9\left(3\Delta^6+8\Delta^5+16\Delta^4+15\Delta^3+11\Delta^2+\Delta-9\right)\Gamma(\Delta)}\{\left(15\Delta^6+64\Delta^5-4\Delta^4-130\Delta^3+244\Delta^2+270\Delta+243\right)C_{0,1,0}$$

$$-12\left(\left(12\Delta^5+9\Delta^4-31\Delta^3+13\Delta^2+34\Delta+24\right)C_{0,1,1}+2\left(-24\Delta^5+27\Delta^4+47\Delta^3-38\Delta^2-17\Delta+15\right)C_{0,2,1}\right)\}\ ,$$

$$t_8 = \frac{-\pi^{7/2}4^{-\Delta-1}\Delta(2\Delta-3)\Gamma\left(\Delta-\frac{3}{2}\right)}{3(\Delta-1)\left(3\Delta^6+8\Delta^5+16\Delta^4+15\Delta^3+11\Delta^2+\Delta-9\right)\Gamma(\Delta)}$$

$$\times\{\left(3\Delta^7+9\Delta^6-8\Delta^5-62\Delta^4-30\Delta^3-190\Delta^2-163\Delta-207\right)C_{0,1,0}$$

$$-12\left(\left(2\Delta^6+\Delta^5-8\Delta^4+2\Delta^3-13\Delta^2-16\Delta-22\right)C_{0,1,1}+2\left(-4\Delta^6+7\Delta^5+4\Delta^4-13\Delta^3+5\Delta^2+8\Delta-7\right)C_{0,2,1}\right)\}\ ,$$

$$t_9 = \frac{\pi^{7/2}4^{-2\Delta-3}C_{0,1,0}}{63\left(\Delta^2-\Delta-1\right)\Gamma(\Delta+4)^2}\{24\sqrt{\pi}(\Delta+2)(\Delta+3)(16\Delta^{12}+112\Delta^{11}+802\Delta^{10}+2041\Delta^9-3583\Delta^8-27783\Delta^7$$

$$-97848\Delta^6-361565\Delta^5-1046943\Delta^4-1943909\Delta^3-2130484\Delta^2-1182496\Delta-72840)\Gamma(2\Delta-2)$$

$$-\frac{4^{\Delta}\Gamma\left(\Delta-\frac{1}{2}\right)\Gamma(\Delta+4)}{\Delta\left(\Delta^2-1\right)}(48\Delta^{12}+560\Delta^{11}+2182\Delta^{10}+2763\Delta^9-7389\Delta^8-69237\Delta^7-307656\Delta^6$$

$$-1103735\Delta^5-3121789\Delta^4-5823663\Delta^3-6399516\Delta^2-3547488\Delta-218520)\}\ .$$

A.15 Details of Spin-4 Calculation in $D > 4$

Constraints From Conservation Equation

Conservation equation leads to 4 relations among the OPE coefficients of $\langle X_{\ell=4} X_{\ell=4} T \rangle$:

$$\tilde{C}_{0,0,0} = \frac{1}{8}\left((d-2)(d+8)\tilde{C}_{1,1,0} - 2d(d+6)\tilde{C}_{0,2,0}\right) + 2\tilde{C}_{0,1,0} \ ,$$
$$\tilde{C}_{0,0,1} = \frac{1}{6}\left(-8d\tilde{C}_{0,2,0} - 2d(d+4)\tilde{C}_{0,2,1} + (d-2)\left((d+6)\tilde{C}_{1,1,1} + 3\tilde{C}_{1,1,0}\right) + 12\tilde{C}_{0,1,1}\right) \ ,$$
$$\tilde{C}_{0,0,2} = \frac{1}{4}\left(-6d\tilde{C}_{0,2,1} - 2d(d+2)\tilde{C}_{0,2,2} + (d-2)\left((d+4)\tilde{C}_{1,1,2} + 2\tilde{C}_{1,1,1}\right) + 8\tilde{C}_{0,1,2}\right) \ ,$$
$$\tilde{C}_{0,0,3} = \frac{1}{2}\left(-4d\tilde{C}_{0,2,2} + (d-2)\left((d+2)\tilde{C}_{1,1,3} + \tilde{C}_{1,1,2}\right) + 4\tilde{C}_{0,1,3}\right) \ .$$

Deriving Constraints from the HNEC

The full expression for $\mathbf{E}(\rho)$ is long and not very illuminating, so we will not transcribe it here. Instead we introduce a new basis $\{\tilde{t}_1, \cdots, \tilde{t}_{12}\}$ in the space of OPE coefficients $\tilde{C}_{i,j,k}$ and use this new basis to derive constraints. The exact structures of $\tilde{t}_1, \cdots, \tilde{t}_{12}$ are not important because the fact that $\tilde{t}_1, \cdots, \tilde{t}_{12}$ are independent linear combinations of $\tilde{C}_{i,j,k}$ is sufficient to rule out the existence of spin-4 operators.

We again start with $\xi = +1$, however, for spin-4, this will be sufficient to rule them out completely. In the limit $\rho \to 1$, the leading contribution to $\mathbf{E}(\rho)$ goes as

$(1-\rho)^{-(d+5)}$, in particular

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_1}{(d-2)d(d+2)(d+4)(1-\rho)^{(d+5)}}\left((d-2)d(d+2)(d+4)\right.$$
$$\left. -32d(d+2)(d+4)\lambda^2 + 288(d+2)(d+4)\lambda^4 - 768(d+4)\lambda^6 + 384\lambda^8\right) + \cdots \ . \tag{A.15.1}$$

Positivity of coefficients of each powers of λ^2 leads to the constraint

$$\tilde{t}_1 = 0 \ . \tag{A.15.2}$$

After imposing this constraint, the next leading term becomes

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_2}{(1-\rho)^{d+4}}\left(1 - \frac{24\lambda^2}{d-2} + \frac{144\lambda^4}{(d-2)d} - \frac{192\lambda^6}{d(d^2-4)}\right) + \cdots \ , \tag{A.15.3}$$

where, positivity now implies

$$\tilde{t}_2 = 0 \ . \tag{A.15.4}$$

After imposing both these constraints the next leading contribution behaves similar to the spin-3 case:

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_3 - (\tilde{a}_3\tilde{t}_3 + \tilde{a}_4\tilde{t}_4)\lambda^2 + \tilde{t}_4\lambda^4 + (\tilde{b}_3\tilde{t}_3 + \tilde{b}_4\tilde{t}_4)\lambda^6 + (\tilde{c}_3\tilde{t}_3 + \tilde{c}_4\tilde{t}_4)\lambda^8}{(1-\rho)^{d+3}} + \cdots \ , \tag{A.15.5}$$

where, $\tilde{a}_3, \tilde{a}_4, \tilde{b}_3, \tilde{b}_4, \tilde{c}_3, \tilde{c}_4$ are numerical factors given later in this appendix. Note that $\tilde{a}_3, \tilde{a}_4 > 0$ and hence positivity of coefficients of λ^0, λ^2 and λ^4 imply that

$$\tilde{t}_3 = \tilde{t}_4 = 0 \ . \tag{A.15.6}$$

The next order contribution has an identical structure:

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_5 - (\tilde{a}_5\tilde{t}_5 + \tilde{a}_6\tilde{t}_6)\lambda^2 + \tilde{t}_6\lambda^4 + (\tilde{b}_5\tilde{t}_5 + \tilde{b}_6\tilde{t}_6)\lambda^6}{(1-\rho)^{d+2}} + \cdots \ , \tag{A.15.7}$$

with $\tilde{a}_5, \tilde{a}_6 > 0$, implying

$$\tilde{t}_5 = \tilde{t}_6 = 0 \ . \tag{A.15.8}$$

So far, everything is very similar to the spin-3 case. But the next order contribution is somewhat different. In the next order, there are three independent structures

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_7 - (\tilde{a}_7\tilde{t}_7 + \tilde{a}_8\tilde{t}_8 + \tilde{a}_9\tilde{t}_9)\lambda^2 + \tilde{t}_8\lambda^4 + \tilde{t}_9\lambda^6 + (\tilde{b}_7\tilde{t}_7 + \tilde{b}_8\tilde{t}_8 + \tilde{b}_9\tilde{t}_9)\lambda^8}{(1-\rho)^{d+1}} + \cdots \ , \tag{A.15.9}$$

where, $\tilde{a}_7, \tilde{a}_8, \tilde{a}_9 > 0$. Positivity now leads to three constraints

$$\tilde{t}_7 = \tilde{t}_8 = \tilde{t}_9 = 0 \ . \tag{A.15.10}$$

However, after imposing these constraints, in the next order we get only two new structures mainly because a lot of contributions vanish after imposing the previous constraints. In particular, we obtain

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_{10} - (\tilde{a}_{10}\tilde{t}_{10} + \tilde{a}_{11}\tilde{t}_{11})\lambda^2 + \tilde{t}_{11}\lambda^4 - (\tilde{b}_{10}\tilde{t}_{10} + \tilde{b}_{11}\tilde{t}_{11})\lambda^6}{(1-\rho)^{d}} + \cdots \tag{A.15.11}$$

with either $\tilde{a}_{10}, \tilde{a}_{11} > 0$ or $\tilde{b}_{10}, \tilde{b}_{11} > 0$ which again implies

$$\tilde{t}_{10} = \tilde{t}_{11} = 0 \ . \tag{A.15.12}$$

Finally, in the next order we get

$$\mathbf{E}_+(\rho) = \frac{\tilde{t}_{12}}{(1-\rho)^{d-1}} \left(1 + \tilde{a}_{12}\lambda^2 + \tilde{b}_{12}\lambda^4 - \tilde{c}_{12}\lambda^6\right) + \cdots \ , \tag{A.15.13}$$

where, $\tilde{a}_{12}, \tilde{b}_{12}, \tilde{c}_{12} > 0$ as shown later in this appendix. Note that unlike the spin-3 case, signs of coefficients of different powers of λ^2 switch sign. Therefore, we can conclude that

$$\tilde{t}_{12} = 0 \ . \tag{A.15.14}$$

$\{\tilde{t}_1, \cdots, \tilde{t}_{12}\}$ forms a complete basis in the space of OPE coefficients and hence the constraints $\tilde{t}_1, \cdots, \tilde{t}_{12} = 0$ necessarily require that all OPE coefficients $\tilde{C}_{i,j,k}$ must vanish implying

$$\langle X_{\ell=4} X_{\ell=4} T \rangle = 0 \ . \tag{A.15.15}$$

$\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ Coefficients

$\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ coefficients are given by

$$\tilde{a}_3 = \frac{2(d(41\Delta+73)-4(\Delta+11))}{(d-2)(d(6\Delta+11)-4(\Delta+3))}\,, \qquad \tilde{a}_4 = \frac{(d-3)d(\Delta+2)}{3(d(6\Delta+11)-4(\Delta+3))}\,,$$
$$\tilde{b}_3 = \frac{48(d(27\Delta+43)+52\Delta+60)}{d\left(d^2-4\right)(d(6\Delta+11)-4(\Delta+3))}\,, \quad \tilde{b}_4 = -\frac{8(d(3\Delta+5)+5\Delta+6)}{(d+2)(d(6\Delta+11)-4(\Delta+3))}\,,$$
$$\tilde{c}_3 = -\frac{192(5\Delta+7)}{d\left(d^2-4\right)(d(6\Delta+11)-4(\Delta+3))}\,, \quad \tilde{c}_4 = \frac{8(2\Delta+3)}{(d+2)(d(6\Delta+11)-4(\Delta+3))}\,,$$
$$\tilde{a}_5 = \frac{d(38\Delta+24)-4(\Delta+8)}{(d-2)(d(4\Delta+3)-2(\Delta+2))}\,, \qquad \tilde{a}_6 = \frac{(d-3)d(\Delta+1)}{3(d(4\Delta+3)-2(\Delta+2))}\,,$$
$$\tilde{b}_5 = \frac{144\Delta}{(d-2)d(d(4\Delta+3)-2(\Delta+2))}\,, \qquad \tilde{b}_6 = \frac{8\Delta+4}{-4d\Delta-3d+2\Delta+4}\,,$$
$$\tilde{a}_7 = \frac{12(3d+1)(\Delta-1)(2\Delta+1)}{(d-2)\left(d\left(13\Delta^2-3\right)+\Delta^2-1\right)}\,, \tilde{b}_7 = \frac{24\Delta\left(\Delta^2-1\right)}{d\left(d^2-3d+2\right)(\Delta+2)\left(d\left(13\Delta^2-3\right)+\Delta^2-1\right)}\,,$$
$$\tilde{a}_8 = \frac{(d-3)d\Delta(2\Delta+1)}{d\left(13\Delta^2-3\right)+\Delta^2-1}\,, \qquad \tilde{b}_8 = \frac{2\left(-4\Delta^3-4\Delta^2+\Delta+1\right)}{(d-1)(\Delta+2)\left(d\left(13\Delta^2-3\right)+\Delta^2-1\right)}\,,$$
$$\tilde{a}_9 = \frac{(d-3)^2 d\Delta(\Delta+1)}{4\left(d\left(13\Delta^2-3\right)+\Delta^2-1\right)}\,, \quad \tilde{b}_9 = -\frac{(2\Delta+1)(d(7\Delta(\Delta+1)-5)-\Delta(\Delta+1))}{(d-1)(\Delta+2)\left(d\left(13\Delta^2-3\right)+\Delta^2-1\right)}\,,$$
$$\tilde{a}_{10} = -\frac{2\left(-((d-9)d+26)\Delta^3-6((d-7)d+4)\Delta^2-(d-1)(11d+2)\Delta+6(d-2)(d+1)\right)}{(d-2)(d-2\Delta)(d(\Delta(\Delta+4)-3)-\Delta(7\Delta+4)+6)}\,,$$
$$\tilde{b}_{10} = -\frac{24\Delta(\Delta+1)(\Delta+2)}{(d-2)(d-2\Delta)\left(d\left(\Delta^2+4\Delta-3\right)-7\Delta^2-4\Delta+6\right)}\,,$$
$$\tilde{a}_{11} = \frac{(d-3)(\Delta-1)(d-2(\Delta+1))}{2(d(\Delta(\Delta+4)-3)-\Delta(7\Delta+4)+6)}\,, \quad \tilde{b}_{11} = \frac{2(\Delta+2)(2\Delta-1)}{d\left(\Delta^2+4\Delta-3\right)-7\Delta^2-4\Delta+6}\,,$$
$$\tilde{a}_{12} = \frac{2\left(2(5-2d)\Delta^2-3d\Delta+d+2\right)}{(d-2)(2\Delta-1)(d-2\Delta+2)}\,, \quad b_{12} = \frac{4\Delta(d(\Delta+3)(2\Delta+1)-2(\Delta(4\Delta+5)+3))}{(d-2)(2\Delta-1)(d-2\Delta)(d-2\Delta+2)}\,,$$
$$\tilde{c}_{12} = \frac{8\Delta(\Delta+1)^2}{(d-2)(2\Delta-1)(d-2\Delta)(d-2\Delta+2)}\,.$$

A.16 Details of CFT_3 calculations

In this appendix, we discuss the details of the the parity even structures for spin 3 operators in $D = 3$. The full expression for $\mathbf{E}(\rho)$ is rather long and not very illuminating, so we will not transcribe it here. Following the logic of the higher D case, we introduce a new basis $\{t_1, \cdots, t_7\}$ in the space of OPE coefficients $C_{i,j,k}$ and use this new basis to derive constraints.

In the limit $\rho \to 1$, the leading parity even contribution to $\mathbf{E}(\rho)$ goes as $(1-\rho)^{-6}$, in particular

$$\mathbf{E}(\rho) = \frac{j_6(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^6} t_1 + \cdots \ , \tag{A.16.1}$$

where, $j_6(\epsilon^{\mu_1\mu_2\mu_3})$ is a specific function of the traceless symmetric polarization tensor. $j_6(\epsilon^{\mu_1\mu_2\mu_3})$ has the property that

$$\begin{aligned} j_6(\epsilon^{\mu_1\mu_2\mu_3}) &\sim \epsilon^{000}\epsilon^{000*} \geq 0 \qquad \text{for} \qquad \epsilon^{\mu_1\mu_2 2} = 0 \ , \\ j_6(\epsilon^{\mu_1\mu_2\mu_3}) &\sim -\epsilon^{222}\epsilon^{222*} \leq 0 \qquad \text{for} \qquad \epsilon^{\mu_1\mu_2 0} = 0 \ . \end{aligned} \tag{A.16.2}$$

Therefore, the HNEC implies that

$$t_1 = 0 \ . \tag{A.16.3}$$

After imposing this constraint, the next leading term becomes

$$\mathbf{E}(\rho) = \frac{j_5(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^5} t_2 + \cdots \ , \tag{A.16.4}$$

where, $j_5(\epsilon^{\mu_1\mu_2\mu_3})$ is another function which has the property that

$$j_5(\epsilon^{\mu_1\mu_2\mu_3}) \sim \text{Re}\left[\epsilon^{000}\left(\epsilon^{001}+\epsilon^{010}+\epsilon^{100}\right)^*\right] \qquad \text{for} \qquad \epsilon^{\mu_1\mu_2 2} = 0 \tag{A.16.5}$$

which changes sign as $\epsilon^{001} \to -\epsilon^{001}$ implying

$$t_2 = 0 \ . \tag{A.16.6}$$

The next order term has two structures:

$$\mathbf{E}(\rho) = \frac{j_4(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^4}t_3 + \frac{\tilde{j}_4(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^4}t_4 + \cdots \ , \tag{A.16.7}$$

where, j_4 and $\tilde{j}_4$ are specific functions of the polarization tensors. Now, applying the HNEC for the following set of polarizations:

$$\begin{aligned}
&(a)\ \epsilon^{000} = \epsilon^{011} = \epsilon^{101} = \epsilon^{110} = 1 \ , \\
&(b)\ \epsilon^{012} = 1 \ , \\
&(c)\ \epsilon^{222} = -\epsilon^{211} = -\epsilon^{121} = -\epsilon^{112} = 1 \ , \\
&(d)\ \epsilon^{000} = \epsilon^{220} = \epsilon^{202} = \epsilon^{022} = 1
\end{aligned} \tag{A.16.8}$$

we find that both t_3 and t_4 must vanish. After imposing these constraints, the next order term also has two structures:

$$\mathbf{E}(\rho) = \frac{j_3(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^3}t_5 + \frac{\tilde{j}_3(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^3}t_6 + \cdots \ , \tag{A.16.9}$$

where, again we will not transcribe j_5 and $\tilde{j}_5$ for simplicity. Now, applying the HNEC for the following set of polarizations:

$$\begin{aligned}
&(a)\ \epsilon^{2\mu\nu} = 0 \ , \\
&(b)\ \epsilon^{012} = \pm 1 \ , \qquad \epsilon^{222} = -\epsilon^{211} = -\epsilon^{121} = -\epsilon^{112} = 1
\end{aligned} \tag{A.16.10}$$

we get

$$t_5 = t_6 = 0 \ . \tag{A.16.11}$$

After imposing all these constraints, we finally obtain

$$\mathbf{E}(\rho) = \frac{j_2(\epsilon^{\mu_1\mu_2\mu_3})}{(1-\rho)^2}t_7 + \cdots \ . \tag{A.16.12}$$

We repeat the same procedure by choosing (a) $\epsilon^{0\mu\nu} = 0$ and (b) $\epsilon^{2\mu\nu} = 0$ that lead to the final constraint

$$t_7 = 0 \ . \tag{A.16.13}$$

Since, $\{t_1, \cdots, t_7\}$ forms a complete basis in the space of OPE coefficients, the constraints $t_1, \cdots, t_7 = 0$ necessarily require that all OPE coefficients $C_{i,j,k}$ must vanish. It is interesting to note that the same set of constraints can also be obtained by using the λ-trick. We can first impose $C_{1,1,k} = 0$ and then use the polarization (4.3.14) to derive constraints in general dimension D. Then taking the limit $D \to 3$ leads to the correct set of constraints at each order.

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